Math 101 Fall 2013
First Homework
Due Wednesday September 25, 2013

1. Recall that if k is a field and $\beta = \{ v_1, \dots, v_n \}$ is a basis for a k-vector space V, then there is a vector space isomorphism $\Phi : V \to k^n$ given by sending $v \in V$ to its coordinate vector $[v]_\beta = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ where the c_i are the unique scalars such that $v = c_1 v_1 + \cdots + c_n v_n$. If W is another k-vector space with basis $\alpha = \{ w_1, \dots, w_m \}$ and if $T : V \to W$ is a linear transformation, then by definition $[T]_\beta^\alpha$ is the $m \times n$ matrix whose j^{th} column is $[Tv_j]_\alpha$. Recall that if $A = (a_{ij})$ is a $m \times n$ matrix and $B = (b_{ij})$ is a $n \times p$ matrix then AB is the $m \times p$ matrix (c_{ij}) with $c_{ij} = \sum_r a_{ir} b_{rj}$. You may want to use the observation (after having checked it without including it in your homework write-up) that the j^{th} column of AB is Ac where c is the j^{th} column of B.

(a) Let V, W, β, α and T be as above. Show that

$$[Tv]_\alpha = [T]_\beta^\alpha [v]_\beta.$$

(b) Suppose that $\gamma = \{ z_1, \dots, z_p \}$ is a basis for a k-vector space Z and that $S : W \to Z$ is linear. Show that

$$[ST]_\beta^\gamma = [S]_\alpha^\gamma [T]_\beta^\alpha.$$

(c) Let $F : \mathbf{R}^2 \to \mathbf{R}^2$ be reflection across the line $y = (\tan\theta)x$. Let $\sigma = \{ e_1, e_2 \}$ be the standard basis for $\mathbf{R}^2$. Find $[F]_\sigma^\sigma$. (I suggest the following. Let $u = (\cos\theta, \sin\theta)$ and $w = (-\sin\theta, \cos\theta)$. Then $\beta = \{ u, w \}$ is a basis for $\mathbf{R}^2$ and since $F(u) = u$ and $F(w) = -w$, the matrix $[F]_\beta^\beta$ has a particularly simple form. But by part (b) above,

$$[F]_\sigma^\sigma = [I]_\beta^\sigma [F]_\beta^\beta [I]_\sigma^\beta$$

where $I : \mathbf{R}^2 \to \mathbf{R}^2$ is the identity map. However one of $[I]_\beta^\sigma$ and $[I]_\sigma^\beta$ is easy to compute and the other is its inverse. For your final answer, you should employ the sum formulas for sin and cos.)

ANS: (a) Let $v = c_1 v_1 + \cdots + c_n v_n$ and let $Tv_j = a_{1j} w_1 + \cdots + a_{mj} w_m$. Then $[T]_\beta^\alpha = (a_{ij})$. On the other hand,

$$\begin{aligned} Tv &= c_1 Tv_1 + \cdots + c_n Tv_n \\ &= c_1 \big(\sum_{k=1}^{m} a_{k1} w_k\big) + \cdots + c_n \big(\sum_{k=1}^{m} a_{kn} w_k\big) \\ &= \big(\sum_{j=1}^{n} a_{1j} c_j\big) w_1 + \cdots + \big(\sum_{j=1}^{n} a_{mj} c_j\big) w_m. \end{aligned}$$

But this just means that

$$\begin{aligned} [Tv]_\beta &= \begin{pmatrix} \sum_{j=1}^n a_{1j} c_j \\ \vdots \\ \sum_{j=1}^n a_{mj} c_j \end{pmatrix} \\ &= (a_{ij}) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \\ &= [T]_\beta^\alpha [v]_\alpha \end{aligned}$$

as required.

(b) The j^{th} column of $[ST]_\beta^\gamma$ is given by $[STv_j]_\gamma$. By part (a), this is just $[S]_\alpha^\gamma [Tv_j]_\alpha$. But $[Tv_j]_\alpha$ is the j^{th} column of $[T]_\beta^\alpha$. Hence, by the remark about matrix multiplication above, $[STv_j]_\gamma$ is the j^{th} column of the product $[S]_\alpha^\gamma [T]_\beta^\alpha$. This suffices.

(c) We use the notation set up in the problem. Clearly $[F]_\beta^\beta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Since $[\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)]_\sigma = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, we also have $[I]_\beta^\alpha = [u\ v] = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ and $[I]_\sigma^\beta = \big([I]_\beta^\alpha\big)^{-1} = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$

Now computing $[F]_\sigma^\sigma$ as suggested in the problem we get

$$[F]_\sigma^\sigma = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

2. Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of sets (a.k.a. a "set of sets", which just sounds awful to me). Let

$$C := \{\, (\alpha, x) \in A \times \bigcup_{\alpha \in A} X_\alpha : x \in X_\alpha \,\}.$$

(Because we can identify X_α with $\{\, (\alpha, x) : x \in X_\alpha \,\}$, C is sometimes called the *disjoint union* of the X_α. For example, think about the case where the X_α are all the same. Then C is quite different from the union.) Show that in the category of sets and functions, the coproducts exist and are given by the disjoint union.

ANS: There are only two things to do here. First we have to equip C with the canonical injections: $i_\alpha : X_\alpha \to C$ given by $i_\alpha(x) = (\alpha, x)$. Then we need to see that C and the i_αs have the required universal property. So let $j_\alpha : X_a \to Y$ be maps. Then we are forced to define $f : C \to Y$ by $f(\alpha, x) = j_\alpha(x)$. This clearly suffices.

3. Let $\mathscr{C}$ be a category in which products and coproducts exist. Recall that $\hom_{\mathscr{C}}(X, Y)$ is a set for any pair of objects in $\mathscr{C}$. Show that there is a unique isomorphism

$$\phi : \hom_{\mathscr{C}}\Big(Y, \prod_{\alpha\in A} X_\alpha\Big) \to \prod_{\alpha\in A} \hom_{\mathscr{C}}(Y, X_\alpha)$$

such that $\pi_\alpha \circ \phi(h) = p_\alpha \circ h$. (Here π_α and p_α are the natural projections for the product in category of sets and maps, and for the product in $\mathscr{C}$, respectively.)

Similarly, show that there is a unique isomorphism

$$\psi : \hom_{\mathscr{C}}\Big(\coprod_{\alpha\in A} X_\alpha, Y\Big) \to \prod_{\alpha\in A} \hom_{\mathscr{C}}(X_\alpha, Y)$$

such that $\pi_\alpha \circ \psi(h) = h \circ i_\alpha$.

ANS: We beat this to death in class.

4. Note that in the category of R-modules, we can think of $\bigoplus_{i=1}^n M_i$ as either the product or the coproduct of the finite set $\{ M_1, \ldots, M_n \}$. Let $\kappa_k : M_k \to \bigoplus_{i=1}^n M_i$ and $\pi_k : \bigoplus_{i=1}^n M_i \to M_k$ be the natural maps. In this instance, question 3 says we can identify the set $\hom\Big(\bigoplus_{i=1}^n M_i, \bigoplus_{j-1}^r N_j\Big)$ with the set $\bigoplus_{i=1,j=1}^{n,r} \hom(M_i, N_j)$; specifically, we identify h with the matrix $[h] = (h_{ij})$ where $h_{ij} = \pi_i \circ h \circ \kappa_j \in \hom(M_j, N_i)$. Thus

$$h(m_1, \ldots, m_n) = \Big(\sum h_{1j}(m_j), \sum h_{2j}(m_j), \ldots, \sum h_{rj}(m_j)\Big)$$

Verify that if $h \in \hom\Big(\bigoplus_{i=1}^n M_i, \bigoplus_{j-1}^r N_j\Big)$ and $k \in \hom\Big(\bigoplus_{j-1}^r N_j, \bigoplus_{k=1}^s P_k\Big)$, then $[k \circ h] = [k][h]$ (with the obvious interpretation of $[k][h]$).

5. Suppose that V and W are finite-dimensional k-vector spaces over the field k. Let $T : V \to W$ be a linear map. Show that there are bases β of V and α of W such that $[T]_\beta^\alpha$ is diagonal (i.e., all off-diagonal entries zero) with diagonal entries in $\{0, 1\}$. (I used the proof of the rank-nullity theorem as a guide.)

ANS: Let $S = \{ w_1, \ldots, w_k \}$ be a basis for the image of T. Let we can extend S to a basis $\alpha = \{ w_1, \ldots, w_m \}$ of W. Let v_i be such that $Tv_i = w_i$ for $1 \le i \le k$. It is a standard linear algebra exercise to show that $\{ v_1, \ldots, v_k \}$ is linearly independent. Hence it too can be extended to a basis $\beta = \{ v_1, \ldots, v_n \}$ of V. Now the matrix $[T]_\beta^\alpha = \left(\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is what we want.

6. Suppose that V is a finite-dimensional k-vector space and that $T : V \to V$ is linear. Show that V has a basis β such that $[T]_\beta^\beta$ is diagonal with entries in $\{0, 1\}$ (as in question 5) if and only if $T = T^2$. Compare with the result from question 5.

ANS: If there is a basis such that $[T]_\beta^\beta$ has the given form, then clearly $([T]_\beta^\beta)^2 = [T]_\beta^\beta$. Hence by problem 1(b), $[T^2]_\beta^\beta = [T]_\beta^\beta$ and $T^2 = T$.

Now suppose that $T = T^2$. Let $\{ w_1, \dots, w_r \}$ be a basis for the image of T. Note that $Tw_s = w_s$ for all s. Let $\{ v_1, \dots, v_r \}$ be a basis for the kernel of T. (Note that s might be zero.) By the rank-nullity theorem, the dimension of V is $r + s$. Therefore to see that $\beta = \{ w_1, \dots, w_r, v_1, \dots, v_s \}$ is a basis for V, it suffices to see that it is linearly independent. So suppose that there are scalars such that

$$c_1 w_1 + \cdots + c_r w_r + d_1 v_1 + \cdots + d_s v_s = 0.$$

Applying T to both sides gives $c_1 w_1 + \cdots + c_r w_r = 0$ and we can conclude each $c_i = 0$. But then we have $d_1 v_1 + \cdots + d_s v_s = 0$. This implies $d_j = 0$ for all j. Hence β is a basis and clearly $[T]_\beta^\beta$ has the required form.

In the previous problem, we were able to choose different bases for the domain and range space. Hence we did not need any conditions on T.

7. Let V and W be k-vector spaces as above. Then $\hom_k(V, W)$ is just a fancy way of describing the set of linear maps from V to W. After picking a bases for V and W, we can identify $\hom_k(V, W)$ with the set $M_{m\times n}(k)$ of $m \times n$ matrices where $m = \dim W$ and $n = \dim V$. We write $\mathrm{GL}_m(k)$ for the invertible $m \times m$-matrices. Recall that A and B in $M_{m\times n}(k)$ are row-equivalent if and only if there is a $P \in \mathrm{GL}_m(k)$ such that $PA = B$ and that each such A is row-equivalent to a unique matrix R in row-reduced echelon form.

(a) Define an equivalence relation on $\hom_k(V, W)$ so that $T \sim S$ if and only if there is an isomorphism $U : W \to W$ such that $S = UT$. If k is a finite field with p elements, $\dim V = 4$ and $\dim W = 2$, then how may equivalence classes are there?

(b) Now define $T \approx S$ if there are isomorphisms $U_1 : V \to V$ and $U_2 : W \to W$ so that $S = U_2 T U_1$. How many $\approx$-equivalence classes are there if $\dim V = n$ and $\dim W = m$?

ANS: Fix bases β for V and α for W. Then $T \mapsto [T]_\beta^\alpha$ establishes a bijection between $\hom_k(V, W)$ and $M_{m\times n}(k)$. Similarly, we get a bijection between the isomorphisms of W with itself and the set $\mathrm{GL}_m(k)$ of invertible $m \times m$-matrices.

(a) Thus on $M_{m\times n}(k)$ our equivalence relation $\sim$ becomes $A \sim B$ if and only if there is a $P \in \mathrm{GL}_m(k)$ such that $A = PB$. That is, A and B are equivalent if and only if they are row equivalent. Therefore by the comments above, the equivalence classes are in one-to-one correspondence with matrices in row-reduced echelon form.

Now if $n = 4$, $m = 2$ and $|k| = p$, then the echelon forms look like

$$\left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \end{smallmatrix}\right)$$
$$\left(\begin{smallmatrix} 0&0&0&1 \\ 0&0&0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&0&1&* \\ 0&0&0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&1&*&* \\ 0&0&0&0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&*&*&* \\ 0&0&0&0 \end{smallmatrix}\right)$$
$$\left(\begin{smallmatrix} 0&0&1&0 \\ 0&0&0&1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&1&*&0 \\ 0&0&0&1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0&1&0&* \\ 0&0&1&* \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&*&*&0 \\ 0&0&0&1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&*&0&* \\ 0&0&1&* \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&0&*&* \\ 0&1&*&* \end{smallmatrix}\right),$$

where a "$*$" can be any one of p elements. Thus there are

$$1 + (1 + p + p^2 + p^3) + (1 + p + p^2 + p^2 + p^3 + p^4) = 3 + 2p + 3p^2 + 2p^3 + p^4$$

classes.

(b) Again, we view $\approx$ and an equivalence relation on $M_{m\times n}(k)$ and then $A \approx B$ if and only if there are invertible matrices P and Q such that $A = PBQ$. It follows from problem , that each equivalence class contains a matrix of the form $\left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Since r is clearly the rank of any matrix in the equivalence class, there is one and only one such matrix in each equivalence class. There are $m+1$ of these (including the zero matrix!), and hence $m+1$ equivalence classes. ERROR: I screwed up. There can be at most $\min\{m, n\}$ ones on the diagonal, so the correct answer is $\min\{m, n\} + 1$ classes.