Dartmouth College

Mathematics 101

Homework 7 (due Thursday, Nov 19)

1. Let $A = \mathbb{Z}$ and $\mathfrak{p} = p\mathbb{Z}$ with p a prime in $\mathbb{Z}$. We have characterized the localization $A_\mathfrak{p} = \mathbb{Z}_\mathfrak{p}$ as $\{a/b \in \mathbb{Q} \mid a, b \in \mathbb{Z},\ p \nmid b,\ \gcd(a, b) = 1\}$.

   (a) Show that every nonzero element in $\mathbb{Z}_\mathfrak{p}$ can be written uniquely as $p^\nu u$ where ν is a nonnegative integer and $u \in \mathbb{Z}_\mathfrak{p}^\times$. You may of course assume unique factorization in $\mathbb{Z}$.

   (b) Characterize all the ideals of $\mathbb{Z}_\mathfrak{p}$, and confirm that $\mathbb{Z}_\mathfrak{p}$ has a unique maximal ideal.

   (c) Show that $\mathbb{Z}_\mathfrak{p}/p\mathbb{Z}_\mathfrak{p} \cong \mathbb{Z}/p\mathbb{Z}$.

2. Let A be a commutative ring with identity.

   (a) Suppose that for each prime ideal $\mathfrak{P}$ in A, the local ring $A_\mathfrak{P}$ has no nonzero nilpotent elements. Show that A has no nonzero nilpotent elements. *Hint:* Show that for an element $x \in A$, the set $\mathrm{Ann}(x) = \{y \in A \mid yx = 0\}$ is an ideal of A. $\mathrm{Ann}(x)$ is called the annihilator of the element x.

   (b) Proof or counterexample: If for each prime $\mathfrak{P}$ of A, each localization $A_\mathfrak{P}$ is an integral domain, then A is an integral domain.

3. Let A be an integral domain, $S \subsetneq A$ a multiplicative subset containing 1 (but not containing 0).

   (a) Show that $S^{-1}A$ is an integral domain.

   (b) Show that if A is a PID (every ideal is principal), so is $S^{-1}A$.

4. Consider the localization of $\mathbb{Z}[x]$ at the prime ideal (x).

   (a) Describe the elements of $\mathbb{Z}[x]_{(x)}$.

   (b) Is (x) maximal in $\mathbb{Z}[x]_{(x)}$? If so, describe the resulting quotient field.

   (c) How does $\mathbb{Z}[x]_{(x)}$ compare to $\mathbb{Q}[x]_{(x)}$?

5. Let A be a commutative ring with identity, and let X be the set of all prime ideals in A. X is called the prime spectrum of A, written *Spec*(A). For each subset $E \subseteq A$, let $V(E)$ denote the set of primes ideals of A which contain E. The properties below

demonstrate that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. This topology is called the Zariski topology on $Spec(A)$.

Prove that:

(a) If $I = \langle E \rangle$ is the ideal generated by E, then $V(I) = V(E)$.

(b) Show that $V(0) = X$ and $V(1) = \emptyset$.

(c) If $\{E_i\}_{i \in I}$ is any family of subsets of A, then $V(\cup_i E_i) = \cap_{i \in I} V(E_i)$.

(d) For any ideals I, J of A, show that $V(I \cap J) = V(IJ) = V(I) \cup V(J)$.