Dartmouth College

Mathematics 101

Homework 3 (due Thursday, Oct 15)

1. Show that the following three statements about finite groups are equivalent. The second is the Feit-Thompson theorem.
   (a) A finite non-abelian simple group has even order.
   (b) A simple group of odd order is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ where p is a prime.
   (c) Every group of odd order is solvable.

2. Let G be a finite group and $H \trianglelefteq G$. Show that G has a composition series one of whose terms is H.

3. Let G be a group and let G' be the subgroup of G generated by the set $\{xyx^{-1}y^{-1} \mid x, y \in G\}$. G' is called the commutator subgroup of G.
   (a) Show that if H is a subgroup of G, then $H \supseteq G'$ if and only if $H \triangleleft G$ and G/H is abelian. In particular, $G' \triangleleft G$ and G/G' is abelian.
   (b) Show that if $\varphi : G \to H$ is a homomorphism of groups, and H is abelian, then φ factors through G/G', that is there is a map $\varphi_* : G/G' \to H$ with $\varphi = \varphi_* \circ \pi$ with $\pi : G \to G/G'$ the standard projection.

4. For a group G, define a sequence of subgroups $G^{(k)}$, by $G^{(1)} = G'$ (the commutator subgroup), and $G^{(k+1)} = [G^{(k)}]'$, the commutator of $G^{(k)}$. Show that G is solvable if and only if $G^{(k)} = \{e\}$ for some $k \geq 1$.

5. Let G be a group, H and K solvable subgroups of G, with $K \triangleleft G$. Show that HK is a solvable subgroup of G.