

# Dartmouth College

Mathematics 101

Homework 6 (due Wednesday, November 14)

1. If  $R$  is a ring with identity, an element  $e \in R$  is called an *idempotent* if  $e^2 = e$ . Notice that aside from  $e = 0, 1$ , all other idempotents are zero divisors. If  $\varphi : R \rightarrow S$  is a homomorphism between rings with identity,  $\varphi(1_R)$  is an idempotent, so to find homomorphisms in which  $\varphi(1_R)$  is not the identity, one must look for idempotents in  $S$ . Find all *ring* homomorphisms  $\varphi : \mathbb{Z}_{120} \rightarrow \mathbb{Z}_{42}$ ; verify they are ring homomorphisms.
2. Let  $R$  be a commutative ring with identity. An element  $x \in R$  is called *nilpotent*, if  $x^n = 0$  for some positive integer  $n$ .
  - (a) Show that the set of nilpotent elements in  $R$  form an ideal, called the *nilradical* of  $R$ .
  - (b) Show that the sum of a unit and a nilpotent element is a unit in  $R$ . *Hint:* As a lemma, show it first with the unit equal to the identity.
3. Let  $R$  be a commutative ring with identity. Let  $R[x]$  be the polynomial ring in one variable with coefficients in  $R$ . Let  $p(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x]$ .
  - (a) Show that  $p$  is nilpotent in  $R[x]$  if and only if  $a_i$  is nilpotent in  $R$  for all  $i \geq 0$ .
  - (b) Show that  $p$  is a unit in  $R[x]$  if and only if  $a_0$  is a unit in  $R$  and the  $a_i$  are nilpotent in  $R$  for  $i \geq 1$ . *Hint:* If  $p = a_0 + \cdots + a_nx^n$  and  $q = b_0 + \cdots + b_mx^m$  with  $pq = 1$ , show by induction on  $r$  that  $a_n^{r+1}b_{m-r} = 0$  to conclude  $a_n$  nilpotent for  $n \geq 1$ .
4. Assume that  $R$  is a commutative ring with identity, and let  $f(x)$  be a monic polynomial in  $R[x]$  of degree  $n \geq 1$ . Let bar notation denote passage to the quotient ring  $R[x]/(f)$ .
  - (a) Show that every element of  $R[x]/(f)$  has a unique representative of the form  $\overline{a_0 + a_1x + \cdots + a_{n-1}x^{n-1}}$  with  $a_i \in R$ .
  - (b) If  $f(x) = x^n - a$  for  $a$  some nilpotent element of  $R$ , show that  $\bar{x}$  is nilpotent in  $R[x]/(f)$ .