Dartmouth College

Mathematics 101

Homework 5 (due Wednesday, October 31)

1. For $n \geq 3$, characterize the center of the symmetric group S_n.

2. Show that for $n \geq 5$, the only normal subgroups of S_n are $\{e\}$, A_n, and S_n. Use this to give an alternate proof to Lang's that S_n is not solvable for $n \geq 5$. This fact is key in showing that the general polynomial of degree $n \geq 5$ is not solvable by radicals.

3. For p and q distinct primes, show that any group of order p^2q is solvable.

4. Semidirect products. We shall show in class that $Aut(\mathbb{Z}_n) \cong \mathbb{Z}_n^\times$.

   (a) Suppose that H_1, H_2 and K are groups, $\sigma : H_1 \to H_2$ is an isomorphism, and $\psi : H_2 \to Aut(K)$ a homomorphism, so that $\varphi = \psi \circ \sigma : H_1 \to Aut(K)$ is also a homomorphism. Show that $K \rtimes_\varphi H_1 \cong K \rtimes_\psi H_2$.

   (b) Suppose that H and K are groups and $\varphi, \psi : H \to Aut(K)$ are monomorphisms with the same image in $Aut(K)$. Show that there exists a $\sigma \in Aut(H)$ such that $\psi = \varphi \circ \sigma$.

   (c) Suppose that H and K are groups, $\varphi, \psi : H \to Aut(K)$ are monomorphisms, and $Aut(K)$ is finite and cyclic. Show that φ and ψ have the same image in $Aut(K)$.

   (d) Let $p < q$ be primes with $p \mid (q-1)$. Let H and K be cyclic groups of order p and q respectively. Let $\varphi, \psi : H \to Aut(K)$ be nontrivial homomorphisms. Observing that $Aut(K)$ is cyclic, show that $K \rtimes_\varphi H \cong K \rtimes_\psi H$.

5. Let $p < q$ be primes, and let G be a group of order pq. We know from class that $G \cong \mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_p$ for some $\varphi : \mathbb{Z}_p \to Aut(\mathbb{Z}_q)$. By analyzing all possible φ, find (up to isomorphism) all groups of order pq. If $p = 2$, describe them without using semidirect products.