Dartmouth College

Mathematics 101

Homework 1 (due Wednesday, October 3)

Let $\mathbb{F}_q$ denote the finite field with q elements.

1. Show that the group $GL_n(\mathbb{F}_q)$ $(n \geq 1)$ has cardinality $(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$. *Hint:* You may use without proof that an $n \times n$ matrix is invertible if and only if its rows (resp. columns) are linearly independent.

2. Find a formula for the number of k-dimensional subspaces of the n-dimensional vector space $\mathbb{F}_q^n$. Prove that the number of k-dimensional subspaces is the same as the number of $(n-k)$-dimensional subspaces. In class, we will consider this duality for finite dimensional vector spaces over an arbitrary field.

3. Let V be an n-dimensional vector space over a field K, and $T : V \to V$ a linear operator such that $T^n = 0$, but $T^{n-1} \neq 0$.

   (a) For each integer j with $1 \leq j \leq n$ show that the rank of T^j is $n - j$.

   (b) Use this fact to show that there is a basis of V with respect to which the matrix of T is strictly upper triangular (i.e. all entries on and below the diagonal are zero).

4. Let V be an n-dimensional vector space over a field K, and $V^* = Hom_k(V, k)$ its dual. For a subspace $W \subset V$, let W^0 be the annihilator of W, that is

$$W^0 = \{\varphi \in V^* \mid \varphi(w) = 0 \text{ for all } w \in W\}.$$

   For subspaces W_1 and W_2 of V, show that

   (a) $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

   (b) $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$.