Dartmouth College

Mathematics 101

Homework 7 (due Wednesday, November 15)

1. Localization.

   (a) Let $A = \mathbb{Z}$ and $\mathfrak{P} = p\mathbb{Z}$ with p a prime in $\mathbb{Z}$. We have characterized the localization $A_{\mathfrak{P}} = \mathbb{Z}_{\mathfrak{P}}$ (the localization of $\mathbb{Z}$ at the prime ideal $\mathfrak{P}$) as $\mathbb{Z}_{\mathfrak{P}} = \{a/b \in \mathbb{Q} \mid a, b \in \mathbb{Z},\ p \nmid b\}$. Show that $\mathbb{Z}_{\mathfrak{P}}/p\mathbb{Z}_{\mathfrak{P}} \cong \mathbb{Z}/p\mathbb{Z}$.

   (b) Let A be a commutative ring with identity.

      i. Suppose that for each prime ideal $\mathfrak{P}$ in A, the local ring $A_{\mathfrak{P}}$ has no nonzero nilpotent elements. Show that A has no nonzero nilpotent elements. *Hint:* Show that for an element $x \in A$, the set $\mathrm{Ann}(x) = \{y \in A \mid yx = 0\}$ is an ideal of A. $\mathrm{Ann}(x)$ is called the annihilator of the element x.

      ii. Proof or counterexample: If each $A_{\mathfrak{P}}$ is an integral domain, then A is an integral domain.

2. Let F be a field, and let $a, b \in F^{\times}$. Denote by $A = \left(\frac{a,b}{F}\right)$ the quaternion algebra over F defined as follows: A is a four-dimensional vector space over F with basis $\{1, i, j, k\}$. The basis elements satisfy $i^2 = a, j^2 = b, ij = k = -ji$, and the scalars in F commute with all elements of A. In fact F is the center of A. The algebra $\mathbb{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$ is known as Hamilton's quaternions.

   (a) There is a natural involution on A denoted $\alpha \mapsto \overline{\alpha}$ which for scalars w, x, y, z takes $\alpha = w + xi + yj + zk$ to $\overline{\alpha} = w - xi - yj - zk$. Define two maps with domain A called the norm and trace, given by $N(\alpha) = \alpha\overline{\alpha}$, and $Tr(\alpha) = \alpha + \overline{\alpha}$.

      i. Find explicit formulas for the norm and trace in terms of the variables w, x, y, z when $\alpha = w + xi + yj + zk$.

      ii. Show that both the norm and trace take values in F, and prove that every element of A is the root of a quadratic equation with coefficients in F.

      iii. If $F = \mathbb{R}$, show that A is a division ring if and only if $a < 0$ and $b < 0$.

   (b) Let R be a ring with identity, and let $\alpha \in R$. Consider the evaluation map $\varphi_\alpha : R[x] \to R$ whose domain is the polynomial ring $R[x]$, defined by $\varphi_\alpha(f) =$

$f(\alpha)$. From class, we know that if R is commutative, then φ_α is a ring homomorphism. Show that if R is not commutative, φ_α is not necessarily a homomorphism. Hint: Hamilton's quaternions would be a nice ring to work with.

(c) Consider the following popular argument in textbooks for showing a nonzero polynomial of degree n with coefficients in a field has at most n distinct roots in the field.

The proof typically proceeds by induction on n. Suppose that A is a field, and let $f(x) \in A[x]$ have degree $n > 1$, and let $\alpha \in A$ with $f(x) = (x - \alpha)g(x)$ for $g \in A[x]$ with degree of g equaling $n - 1$. Let β be a root of f and assume that $\alpha \neq \beta$. Then β is a root of g, and so by induction f has at most n distinct roots.

While the argument can be made rigorous in the case A is a field, it is rarely done. Given the exact argument as above, let A be a division ring (necessarily with identity). Find a counterexample to the assertion about the number of distinct roots, and explain where the gap appears in the above argument in the case of a non-commutative division ring.