

Dartmouth College

Mathematics 101

Homework 5 (due Wednesday, October 25)

1. Denote by $Aut(\mathbb{Z}_n)$ the group of automorphisms of \mathbb{Z}_n (viewing \mathbb{Z}_n as an additive group). Show that $Aut(\mathbb{Z}_n) \cong \mathbb{Z}_n^\times$ (\mathbb{Z}_n^\times the multiplicative group of the ring \mathbb{Z}_n). It may be of use to recall some of the work we did early in the term on homomorphisms with domain \mathbb{Z}_n .
2. Semidirect products.
 - (a) Suppose that H_1, H_2 and K are groups, $\sigma : H_1 \rightarrow H_2$ is an isomorphism, and $\psi : H_2 \rightarrow Aut(K)$ a homomorphism, so that $\varphi = \psi \circ \sigma : H_1 \rightarrow Aut(K)$ is also a homomorphism. Show that $K \rtimes_{\varphi} H_1 \cong K \rtimes_{\psi} H_2$.
 - (b) Suppose that H and K are groups and $\varphi, \psi : H \rightarrow Aut(K)$ are monomorphisms with the same image in $Aut(K)$. Show that there exists a $\sigma \in Aut(H)$ such that $\psi = \varphi \circ \sigma$.
 - (c) Suppose that H and K are groups, $\varphi, \psi : H \rightarrow Aut(K)$ are monomorphisms, and $Aut(K)$ is finite and cyclic. Show that φ and ψ have the same image in $Aut(K)$.
 - (d) Let $p < q$ be primes with $p \mid (q-1)$. Let H and K be cyclic groups of order p and q respectively. Let $\varphi, \psi : H \rightarrow Aut(K)$ be nontrivial homomorphisms. Observing that $Aut(K)$ is cyclic, show that $K \rtimes_{\varphi} H \cong K \rtimes_{\psi} H$.
3. Let $p < q$ be primes, and let G be a group of order pq . We know from class that $G \cong \mathbb{Z}_q \rtimes_{\varphi} \mathbb{Z}_p$ for some $\varphi : \mathbb{Z}_p \rightarrow Aut(\mathbb{Z}_q)$. By analyzing all possible φ , find (up to isomorphism) all groups of order pq . If $p = 2$, describe them without using semidirect products.
4. (Proof or counterexample) Consider the validity of a cancellation law for direct products: Let G, H, K be finitely generated abelian groups and suppose that $G \times H \cong G \times K$. Then $H \cong K$. (I asked an unrestricted version of this question on the first homework set)