Dartmouth College

Mathematics 101

Homework 4 (due Wednesday, October 18)

1. For a group G, $\mathrm{Tor}(G) = \{g \in G \mid g^n = e \text{ for some } n \geq 1\}$ is called the set of *torsion* elements of G. Of course this really is only interesting for infinite groups.

   (a) If G is abelian, show that $\mathrm{Tor}(G)$ is a subgroup of G, called its torsion subgroup.

   (b) If G is not abelian, show that $\mathrm{Tor}(G)$ need not be a subgroup of G. One can find a nice counterexample in $G = SL_2(\mathbb{Z}) = \langle S, T \rangle$ where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Hint: ST is a nice element.

2. For $n \geq 3$, characterize the center of the symmetric group S_n.

3. For $n \geq 5$, show that the only normal subgroups of S_n are $\{e\}$, A_n, and S_n. Use this to show that S_n is not solvable for $n \geq 5$. This fact is key in showing that the general polynomial of degree $n \geq 5$ is not solvable by radicals.

4. Let $p < q$ be primes and G a nonabelian group of order pq. Show that there is an embedding of G into the symmetric group S_q.