Dartmouth College

Mathematics 101

Homework 3 (due Wednesday, Oct 11)

1. Let G be a group, and H a normal subgroup. Show that G is solvable if and only if H and G/H are solvable.

2. Show that the following three statements are equivalent. The second is the Feit-Thompson theorem.

   (a) A finite non-abelian simple group has even order.

   (b) A simple group of odd order is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ where p is a prime.

   (c) Every group of odd order is solvable.

3. Let G be a finite group and $H \trianglelefteq G$. Show that G has a composition series one of whose terms is H.

4. Let G be a group and let G' be the subgroup of G generated by the set $\{xyx^{-1}y^{-1} \mid x, y \in G\}$. G' is called the commutator subgroup of G.

   (a) Show that if H is a subgroup of G, then $H \supseteq G'$ if and only if $H \trianglelefteq G$ and G/H is abelian. In particular, $G' \trianglelefteq G$ and G/G' is abelian.

   (b) Show that if $\varphi : G \to H$ is a homomorphism of groups, and H is abelian, then φ factors through G/G', that is there is a map $\varphi_* : G/G' \to H$ with $\varphi = \varphi_* \circ \pi$ with $\pi : G \to G/G'$ the standard projection.

5. Let G be a finite group and H a proper subgroup. Show that G is not the union of conjugates of H.