Dartmouth College

Mathematics 101

Homework 1 (due Wednesday, Sept 27)
(Some warm-up exercises)

1. Let S be a nonempty set. Show that there is a one-to-one correspondence between the partitions of S and equivalence relations on S. Note this is essentially proposition 2 of Chapter 0 (the proof of which is left as an exercise).

2. Without quoting a formula from the book, show that if F is a finite field with q elements, then the group $GL_n(F)$ ($n \geq 1$) has cardinality $(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$. *Hint:* You may use without proof that an $n \times n$ matrix is invertible if and only if its rows (resp. columns) are linearly independent.

3. (Proof or counterexample) Let G be a finite group of order n. Then G has at most finitely many isomorphism types, that is there is a finite set of groups, so that G is necessarily isomorphic to one in the finite set. If you decide the number is finite, can you give an upper bound in terms of n?

4. (Proof or counterexample) Consider the validity of a cancellation law for direct products: Let G, H, K be groups and suppose that $G \times H \cong G \times K$. Then $H \cong K$.

5. Let $G = \{\zeta \in \mathbb{C} \mid \zeta^n = 1 \text{ for some } n \geq 1\}$, that is the set of roots of unity in $\mathbb{C}$. Verify that G is a group. Let $k > 1$ be a fixed integer and consider the map $\zeta \mapsto \zeta^k$. Show that this map is a surjective homomorphism from G to G, but not an isomorphism.