Dartmouth College

Mathematics 101

Homework 5 (due Wednesday, October 27)

1. Denote by $Aut(\mathbb{Z}_n)$ the group of automorphisms of $\mathbb{Z}_n$ (viewing $\mathbb{Z}_n$ as an additive group). Show that $Aut(\mathbb{Z}_n) \cong \mathbb{Z}_n^\times$ ($\mathbb{Z}_n^\times$ the multiplicative group of the ring $\mathbb{Z}_n$). It may be of use to recall some of the work we did early in the term on homomorphisms with domain $\mathbb{Z}_n$.

2. Semidirect products.

   (a) Suppose that H_1, H_2 and K are groups, $\sigma : H_1 \to H_2$ is an isomorphism, and $\psi : H_2 \to Aut(K)$ a homomorphism, so that $\varphi = \psi \circ \sigma : H_1 \to Aut(K)$ is also a homomorphism. Show that $K \rtimes_\varphi H_1 \cong K \rtimes_\psi H_2$.

   (b) Suppose that H and K are groups and $\varphi, \psi : H \to Aut(K)$ are monomorphisms with the same image in $Aut(K)$. Show that there exists a $\sigma \in Aut(H)$ such that $\psi = \varphi \circ \sigma$.

   (c) Suppose that H and K are groups, $\varphi, \psi : H \to Aut(K)$ are monomorphisms, and $Aut(K)$ is finite and cyclic. Show that φ and ψ have the same image in $Aut(K)$.

   (d) Let $p < q$ be primes with $p \mid (q-1)$. Let H and K be cyclic groups of order p and q respectively. Let $\varphi, \psi : H \to Aut(K)$ be nontrivial homomorphisms. Observing that $Aut(K)$ is cyclic, show that $K \rtimes_\varphi H \cong K \rtimes_\psi H$.

3. Let $p < q$ be primes, and let G be a group of order pq.

   (a) Show that if $p \nmid (q-1)$, then G is cyclic.

   (b) Show that if $p \mid (q-1)$, then either G is cyclic or $G \cong \mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_p$ for some (and hence any) nontrivial $\varphi : \mathbb{Z}_p \to Aut(\mathbb{Z}_q)$. In particular, if $p = 2$, show that $G \cong \mathbb{Z}_{2q}$ or D_{2q}, the dihedral group.

4. For a group G, $\text{Tor}(G) = \{g \in G \mid g^n = e \text{ for some } n \geq 1\}$ is called the set of *torsion* elements of G. Of course this really is only interesting for infinite groups.

   (a) If G is abelian, show that $\text{Tor}(G)$ is a subgroup of G, called its torsion subgroup.

(b) If G is not abelian, show that $\mathrm{Tor}(G)$ need not be a subgroup of G. One can find a nice counterexample in $G = SL_2(\mathbb{Z}) = \langle S, T \rangle$ where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Hint: ST is a nice element.