Dartmouth College

Mathematics 101

Homework 1 (due Wednesday, Sept 29)

1. Let G be a group and H, K subgroups of G. The *normalizer* of H in G, denoted in Lang by N_H (with G implicit), is defined by $N_H = \{g \in G \mid gHg^{-1} = H\}$. In other books you might see this written as $N_G(H)$.

   Show that if $K \subseteq N_H$, then $HK = KH$ is a subgroup of G, and $H \triangleleft KH$. Of course in particular, if $H \triangleleft G$, then HK is a subgroup of G.

2. Let G be a group, and $H_1, H_2, \ldots, H_n$ subgroups of G. Suppose that

   (a) $G = H_1 H_2 \cdots H_n$

   (b) $H_{i+1} \cap (H_1 H_2 \cdots H_i) = \{e\}$ for all $1 \leq i \leq n-1$.

   (c) The elements of H_i commute with those of H_j for all $i \neq j$.

   Show that the map $\varphi : H_1 \times \cdots \times H_n \to G$ given by $\varphi((h_1, \ldots, h_n)) = h_1 h_2 \cdots h_n$ is an isomorphism.

3. In the previous problem show that condition (c) can be replaced with the following:

   (c') $H_i \triangleleft G$ for $1 \leq i \leq n$

   Note: it is not the case that (c) and (c') are equivalent, however, you can show that (a), (b), (c) are equivalent to (a), (b), (c').

   As we remarked in class, condition (b) is often replaced by the symmetric, but more stringent condition that $H_i \cap \langle \cup_{i \neq j} H_j \rangle = \{e\}$.