# Dartmouth College

Mathematics 101

Homework 6 (due Wednesday, November 19)

1. Let $A = \mathbb{Z}$ and $\mathfrak{p} = p\mathbb{Z}$ with $p$ a prime in $\mathbb{Z}$. We have characterized the localization $A_\mathfrak{p} = \mathbb{Z}_\mathfrak{p}$ as $\{a/b \in \mathbb{Q} \mid a, b \in Z,\ p \nmid b,\ \gcd(a, b) = 1\}$.

   (a) Characterize the unit group $\mathbb{Z}_\mathfrak{p}^\times$.

   (b) Show that every nonzero element in $\mathbb{Z}_\mathfrak{p}$ can be written uniquely as $p^\nu u$ where $\nu$ is a nonnegative integer and $u \in \mathbb{Z}_\mathfrak{p}^\times$. You may of course assume unique factorization in $\mathbb{Z}$.

   (c) Characterize all the ideals of $\mathbb{Z}_\mathfrak{p}$, and confirm that $\mathbb{Z}_\mathfrak{p}$ has a unique maximal ideal.

   (d) Show that $\mathbb{Z}_\mathfrak{p}/p\mathbb{Z}_\mathfrak{p} \cong \mathbb{Z}/p\mathbb{Z}$.

2. Let A be a ring with identity, and let $\alpha \in A$. Consider the evaluation map $\varphi_\alpha : A[x] \to A$ whose domain is the polynomial ring $A[x]$, defined by $\varphi_\alpha(f) = f(\alpha)$.

   (a) If $A$ is commutative, show that $\varphi_\alpha$ is a ring homomorphism.

   (b) If $A$ is not commutative, give a counterexample. Note: Hamilton's quaternions, defined on page 117 of your text, is a very nice ring. Also, while we have not yet formally defined polynomial rings yet, I have confidence you'll do the right thing.

3. Consider the following popular argument in textbooks for showing a nonzero polynomial of degree $n$ with coefficients in a field has at most $n$ distinct roots in the field.

   The proof typically proceeds by induction on $n$. Suppose that $A$ is a field, and let $f(x) \in A[x]$ have degree $n > 0$, and let $\alpha \in A$ with $f(x) = (x - \alpha)g(x)$ for $g \in A[x]$ with degree of $g$ equaling $n - 1$. Let $\beta$ be a root of $f$ and assume that $\alpha \neq \beta$. Then $\beta$ is a root of $g$, and so by induction $f$ has at most $n$ distinct roots.

   While the argument can be made rigorous in the case $A$ is a field, it is rarely done. Given the exact argument as above, let $A$ be a division ring (necessarily with identity). Find a counterexample to the assertion about the number of distinct roots, and explain where there is a gap in the argument in the case of a non-commutative division ring.

4. Let $A$ be an integral domain, and $T \subset S$ two multiplicative subsets of $A$, with $0 \notin S$. Show that there is a natural embedding of $T^{-1}A \to S^{-1}A$.