Dartmouth College

Mathematics 101

Homework 4 (due Wednesday, October 22)

1. For a group G, $\text{Tor}(G) = \{g \in G \mid g^n = e \text{ for some } n \geq 1\}$ is called the set of *torsion* elements of G. Of course this really is only interesting for infinite groups.

   (a) If G is abelian, show that $\text{Tor}(G)$ is a subgroup of G, called its torsion subgroup.

   (b) If G is not abelian, show that $\text{Tor}(G)$ need not be a subgroup of G. One can find a nice counterexample in $G = SL_2(\mathbb{Z}) = \langle S, T \rangle$ where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Hint: ST is a nice element.

2. For $n \geq 3$, characterize the center of the symmetric group S_n.

3. For $n \geq 5$, show that the only normal subgroups of S_n are $\{e\}$, A_n, and S_n. This result is important in Galois theory when you want to show that the general polynomial of degree $n \geq 5$ is not "solvable by radicals".

4. Let H be a group. By H^n we mean the direct product of H with itself n times, that is the set H^n endowed with componentwise operations. Show that S_n acts on H^n via $(\sigma, (h_1, \ldots, h_n)) \mapsto (h_{\sigma^{-1}(1)}, \ldots, h_{\sigma^{-1}(n)})$. **N.B.** The obvious map $(\sigma, (h_1, \ldots, h_n)) \mapsto (h_{\sigma(1)}, \ldots, h_{\sigma(n)})$ is **not** a (left) action.

   Hint: This can be a bit subtle with notation. You may find the following observation useful. In set theory, Y^X denotes the set of functions $f : X \to Y$, so one can interpret H^n as H^X where $X = \{1, \ldots, n\}$. Now show that the natural action of S_n on X induces an action of S^n on $H^X = H^n$ as suggested above.