TRACE FORMULAS FOR THE NORM ONE GROUP OF TOTALLY DEFINITE QUATERNION ALGEBRAS

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WITH AN APPENDIX BY JOHN VOIGHT

Abstract. In his pioneering work [Crelle's Journal, 1955], Eichler established the theory of trace formulas for Brandt matrices of quaternion orders. From it he derived a class number formula for Eichler orders in a totally definite quaternion algebra D. Extending Eichler's work, Pizer [Crelle's Journal, 1973] proved a formula for the type number of Eichler orders in D. In this paper, we extend their results to the norm one group of D. More precisely, we present a class number formula for the norm one group of D with respect to a class of orders $\mathcal{O}$, called *residually unramified orders*, which includes all Eichler orders. Our second result gives a formula for the number of ideal classes in the spinor class of $\mathcal{O}$, which refines Eichler's class number formula. It is worth mentioning that these class number formulas not only depend on the genus of orders as Eichler and Pizer's formulas, but also depend on the orders themselves. We introduce certain auxiliary invariants in order to keep track of the global information on the relationship between certain CM orders and $\mathcal{O}$, and use them to describe our formulas. Both our class number formulas make use of the optimal spinor selectivity theory for quaternion orders.

1. Introduction

Let F be a totally real number field, and D be a *totally definite* quaternion F-algebra, that is, $D \otimes_{F,\sigma} \mathbb{R}$ is isomorphic to the Hamilton quaternion algebra $\mathbb{H}$ for every embedding $\sigma : F \hookrightarrow \mathbb{R}$. Let O_F be the ring of integers of F, and $\mathcal{O}$ be an O_F-order (of full rank) in D. By definition, the class number $h(\mathcal{O})$ is the cardinality of the finite set $\mathrm{Cl}(\mathcal{O})$ of locally principal right $\mathcal{O}$-ideal classes in D. If we write $\widehat{\mathcal{O}}$ for the profinite completion of $\mathcal{O}$, and $\widehat{D}$ for the ring of finite adeles of D, then

$$h(\mathcal{O}) = |\mathrm{Cl}(\mathcal{O})| = |D^\times \backslash \widehat{D}^\times / \widehat{\mathcal{O}}^\times|. \tag{1.1}$$

It is well known that $h(\mathcal{O})$ depends only on the genus of $\mathcal{O}$. In other words, $h(\mathcal{O}') = h(\mathcal{O})$ for any other order $\mathcal{O}'$ *in the same genus* as $\mathcal{O}$, that is, there exists $x \in \widehat{D}^\times$ such that $\widehat{\mathcal{O}}' = x\widehat{\mathcal{O}}x^{-1}$. The class number $h(\mathcal{O})$ can be computed by the Eichler class number formula [40, Corollaire V.2.5, p. 144] (see also [21, Theorem 2], [41, Theorem 30.8.6] and [44, Theorem 1.5]). To state this formula, we set up some notations related to certain quadratic O_F-orders called *CM O_F-orders*.

Since D is totally definite, a quadratic field extension K/F embeds into D only if K/F is a CM-extension (that is, a totally imaginary quadratic extension of the totally real field F). An O_F-order B of full rank in a CM-extension of F will be called a *CM O_F-order*. Let

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$h(B)$ be the class number of B, and $w(B)$ be the unit group index $[B^\times : O_F^\times]$. According to [31, Remarks, p. 92] (cf. [23, §3.1] and [44, §3.3]) there are only finitely many CM O_F-orders B satisfying $w(B) > 1$, so we collect them into a finite set $\mathscr{B}$. For each finite prime $\mathfrak{p}$ of F, we write $B_\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ for the $\mathfrak{p}$-adic completions of B and $\mathcal{O}$ respectively. Let $\mathrm{Emb}(B, \mathcal{O})$ be the set of *optimal embeddings* of B into $\mathcal{O}$, that is,

$$\mathrm{Emb}(B, \mathcal{O}) := \{\varphi \in \mathrm{Hom}_F(K, D) \mid \varphi(K) \cap \mathcal{O} = \varphi(B)\}, \text{ where } K = \mathrm{Frac}(B). \tag{1.2}$$

The unit group $\mathcal{O}^\times$ acts on $\mathrm{Emb}(B, \mathcal{O})$ from the right by $\varphi \mapsto u^{-1}\varphi u$ for all $\varphi \in \mathrm{Emb}(B, \mathcal{O})$ and $u \in \mathcal{O}^\times$. The number of orbits is finite both in the global and local cases, so we put

$$m(B, \mathcal{O}, \mathcal{O}^\times) := |\mathrm{Emb}(B, \mathcal{O})/\mathcal{O}^\times|, \quad m(B_\mathfrak{p}, \mathcal{O}_\mathfrak{p}, \mathcal{O}_\mathfrak{p}^\times) := |\mathrm{Emb}(B_\mathfrak{p}, \mathcal{O}_\mathfrak{p})/\mathcal{O}_\mathfrak{p}^\times|. \tag{1.3}$$

Following [40, §V.2, p. 143], we write

$$M(B) := \frac{h(B)}{w(B)} \prod_{\mathfrak{p}} m_\mathfrak{p}(B), \tag{1.4}$$

where the product runs over all finite primes $\mathfrak{p}$ of F, and $m_\mathfrak{p}(B) := m(B_\mathfrak{p}, \mathcal{O}_\mathfrak{p}, \mathcal{O}_\mathfrak{p}^\times)$. The product is well-defined since $m_\mathfrak{p}(B) = 1$ for almost all $\mathfrak{p}$ ([40, Theorem II.3.2]).

Theorem 1.1 (Eichler class number formula)**.** *Let $F, D, \mathcal{O}$ be as above. Then*

$$h(\mathcal{O}) = \mathrm{Mass}(\mathcal{O}) + \frac{1}{2} \sum_{B \in \mathscr{B}} (w(B) - 1) M(B).$$

Here $\mathrm{Mass}(\mathcal{O})$ *denotes the mass of $\mathcal{O}$ as defined in* (3.1)*, and it can be computed by Köner's formula (3.2).*

As the name suggests, this formula is first derived by Eichler himself [16] for Eichler orders of square-free level in a definite quaternion $\mathbb{Q}$-algebra. It is further generalized by Pizer [32], Vignéras [40], Körner [21] and many others. In terms of algebraic groups, the Eichler class number formula can be regarded as a formula for the class number of the multiplicative group G of D over F (with respect to a suitable open compact subgroup of $G(\widehat{F})$). Similarly, the type number formulas of Pizer [31] and Körner [21] can be interpreted as class number formulas for the adjoint group G^{ad}. In this paper, we extend such formulas to the class number formula for the derived group G^{der} and its variant. More explicitly, we present two class number formulas for the following two quantities:

$$h^1(\mathcal{O}) := |D^1 \backslash \widehat{D}^1 / \widehat{\mathcal{O}}^1|, \qquad h_{\mathrm{sc}}(\mathcal{O}) := |D^\times \backslash \big(D^\times \widehat{D}^1 \widehat{\mathcal{O}}^\times\big) / \widehat{\mathcal{O}}^\times|. \tag{1.5}$$

Here for any set $X \subseteq \widehat{D}$, we put $X^1 := \{x \in X \mid \mathrm{Nr}(x) = 1\}$, where $\mathrm{Nr} : \widehat{D} \to \widehat{F}$ denotes the reduced norm map. In particular, $\widehat{D}^1 = \ker(\widehat{D}^\times \xrightarrow{\mathrm{Nr}} \widehat{F}^\times) = G^{\mathrm{der}}(\widehat{F})$. There is a canonical surjection $D^1 \backslash \widehat{D}^1 / \widehat{\mathcal{O}}^1 \twoheadrightarrow D^\times \backslash (D^\times \widehat{D}^1 \widehat{\mathcal{O}}^\times) / \widehat{\mathcal{O}}^\times$, so we always have $h^1(\mathcal{O}) \geq h_{\mathrm{sc}}(\mathcal{O})$. The main difference between computing $h(\mathcal{O})$ (or the type number $t(\mathcal{O})$ of $\mathcal{O}$) and $h^1(\mathcal{O})$ (or $h_{\mathrm{sc}}(\mathcal{O})$) is that the class number $h(\mathcal{O})$ depends only on the genus of $\mathcal{O}$ while the latter does not in general. Thus for $h^1(\mathcal{O})$ and $h_{\mathrm{sc}}(\mathcal{O})$, the input datum $\mathcal{O}$ is not purely local and the output class number formulas would be expected to include global information of $\mathcal{O}$.

Note that if D is indefinite (that is, unramified at some infinite place of F) rather than totally definite as we have assumed, then $h^1(\mathcal{O}) = h_{\mathrm{sc}}(\mathcal{O}) = 1$ by the strong approximation theorem ([27, Theorem 7.7.5] or [40, Theorem III.4.3]). This is precisely the reason why we focus only on the totally definite case.

The first class number $h^1(\mathcal{O})$ can be interpreted as follows. Let V be an n-dimensional right vector space over D, and $\psi : V \times V \to D$ be a positive definite quaternion Hermitian form with respect to the canonical involution $x \mapsto \bar{x}$ of D. By convention, ψ is D-linear in its second variable and anti-linear in its first variable. A classical result of Shimura [37, §2.2] shows that there exists an identification $V = D^n$ such that ψ takes the form

$$\psi : D^n \times D^n \to D \qquad (x, y) \mapsto \sum_{i=1}^{n} \bar{x}_i y_i. \tag{1.6}$$

Let $G := U_n(D, \psi)$ be the unitary group with respect to ψ as above. Given a right $\mathcal{O}$-lattice L in V, we write $\widehat{L}$ for its profinite completion $L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. Two $\mathcal{O}$-lattices L and L' in V are said to be *isometric* if there exists $g \in G(F)$ such that $L' = gL$, and they are said to be in the *same genus* if there exists $x \in G(\widehat{F})$ such that $\widehat{L}' = x\widehat{L}$. The number of isometric classes of $\mathcal{O}$-lattices in the genus of L is called the *class number of L with respect to ψ* and will be denoted by $h(L, \psi)$. If we write $\mathrm{Stab}(\widehat{L})$ for the stabilizer of $\widehat{L}$ in $G(\widehat{F})$, then $h(L, \psi) = |G(F) \backslash G(\widehat{F}) / \mathrm{Stab}(\widehat{L})|$. When $n = 1$, the group G coincides with the reduced norm one group $\underline{D}^1$ of D (regarded as a linear algebraic group over F), L is a fractional right $\mathcal{O}$-ideal in D, and $\mathrm{Stab}(\widehat{L}) = \widehat{\mathcal{O}_l(L)} \cap \widehat{D}^1$, where $\mathcal{O}_l(L)$ denotes the *left order* of L:

$$\mathcal{O}_l(L) := \{\alpha \in D \mid \alpha L \subseteq L\}. \tag{1.7}$$

Thus in this case $h(L, \psi) = h^1(\mathcal{O}_l(L))$.

The computation of the *class number* $h(L, \psi)$ is a classical problem that has been studied by many people. Suppose for the moment that $F = \mathbb{Q}$ and $\mathcal{O}$ is a maximal order in D. If $\dim_D V = 1$, then the canonical map $D^1 \backslash \widehat{D}^1 / \widehat{\mathcal{O}}^1 \to D^\times \backslash \widehat{D}^\times / \widehat{\mathcal{O}}^\times$ is a bijection because both the narrow class group and the positive unit group of $\mathbb{Z}$ are trivial, so $h^1(\mathcal{O}) = h(\mathcal{O})$ in this case. In the 2-dimensional case, Hashimoto and Ibukiyama [18] obtained class number formulas for arbitrary genera of maximal $\mathcal{O}$-lattices (see [37, §2.3] for the notion of maximal lattices). Hashimoto [17] also obtained a class number formula for the principal genus (that is, $L = \mathcal{O}^3$) in dimension 3 under the further assumption that the discriminant of D is a prime number. Suppose now that F is an arbitrary totally real field and keep the assumption that $\mathcal{O}$ is maximal. In his habilitation thesis, Kirschmer [20, §9] gives a complete classification of all definite quaternion hermitian lattices with class number at most two using computer algorithms. For example, in the 1-dimensional case, he lists 69 totally definite quaternion algebras over 29 different base fields F that admit a hermitian lattice of class number one. Independently using a different method, Ibukiyama, Karemaker and Yu [19] also give a complete list of maximal $\mathcal{O}$-lattices of class number one in the case $F = \mathbb{Q}$.

Now return to the general case where F and $\mathcal{O}$ are both arbitrary. For the meaning of $h_{\mathrm{sc}}(\mathcal{O})$, we recall the following notions from [9, §1].

Definition 1.2. Two O_F-orders $\mathcal{O}$ and $\mathcal{O}'$ in D are in the *same spinor genus* (and denoted by $\mathcal{O} \sim \mathcal{O}'$) if there exists $x \in D^\times \widehat{D}^1$ such that $\widehat{\mathcal{O}}' = x\widehat{\mathcal{O}}x^{-1}$. Similarly, two locally principal right $\mathcal{O}$-ideals I and I' are in the *same spinor class* if there exists $x \in D^\times \widehat{D}^1$ such that $\widehat{I}' = x\widehat{I}$.

Note that every genus of O_F-orders consists of finitely many spinor genera because any two O_F-orders having the same type lie in the same spinor genus and the type number is finite. It is a straightforward exercise to check that both $h^1(\mathcal{O})$ and $h_{\mathrm{sc}}(\mathcal{O})$ depend only on the spinor genus of $\mathcal{O}$. If we denote by $\mathrm{Cl}_{\mathrm{sc}}(\mathcal{O})$ the set of locally principal right $\mathcal{O}$-ideal classes within the spinor class of $\mathcal{O}$ itself, then $h_{\mathrm{sc}}(\mathcal{O}) = |\mathrm{Cl}_{\mathrm{sc}}(\mathcal{O})|$, so the subscript $_{\mathrm{sc}}$ in $h_{\mathrm{sc}}(\mathcal{O})$ stands for "spinor class". By definition, $\mathrm{Cl}_{\mathrm{sc}}(\mathcal{O})$ is a subset of $\mathrm{Cl}(\mathcal{O})$. It will be shown in (2.13) that $h(\mathcal{O})$ can be expressed as a finite sum of several $h_{\mathrm{sc}}(\mathcal{O}')$ for various orders $\mathcal{O}'$, so a class number formula for $h_{\mathrm{sc}}(\mathcal{O})$ is naturally a refinement of the Eichler class number formula for $h(\mathcal{O})$.

In a special case, the class number $h_{\mathrm{sc}}(\mathcal{O})$ can also be interpreted as follows. Let $\mathrm{Cl}^+(O_F)$ be the narrow class group of O_F. There is a canonical surjective map $\mathrm{Nr} : \mathrm{Cl}(\mathcal{O}) \to \mathrm{Cl}^+(O_F)$ that sends each locally principal right $\mathcal{O}$-ideal class $[I]$ to the narrow O_F-ideal class $[\mathrm{Nr}(I)]_+$. If $\mathrm{Nr}(\widehat{\mathcal{O}}^\times) = \widehat{O}_F^\times$ (e.g. if $\mathcal{O}$ is an Eichler order), then two locally principal right $\mathcal{O}$-ideals I and I' are in the same spinor class if and only if their reduced norms $\mathrm{Nr}(I)$ and $\mathrm{Nr}(I')$ belong to the same narrow O_F-ideal class. In this case, $\mathrm{Cl}_{\mathrm{sc}}(\mathcal{O})$ is the neutral fiber of $\mathrm{Nr} : \mathrm{Cl}(\mathcal{O}) \to \mathrm{Cl}^+(O_F)$, and $h_{\mathrm{sc}}(\mathcal{O})$ measures its cardinality. From (2.12), the cardinality of any other fiber is also given by $h_{\mathrm{sc}}(\mathcal{O}')$ for a suitable $\mathcal{O}'$.

To state our formulas for $h^1(\mathcal{O})$ and $h_{\mathrm{sc}}(\mathcal{O})$, we need some more notations. For each CM O_F-order B, let $\boldsymbol{\mu}(B)$ be the group of roots of unity in B. We put

$$\mathscr{B}^1 := \{B \mid B \text{ is a CM } O_F\text{-order with } |\boldsymbol{\mu}(B)| > 2\}. \tag{1.8}$$

Note that $\mathscr{B}^1$ is a subset of the finite set $\mathscr{B}$ appearing in the Eichler class number formula, so it is finite as well. Given a CM O_F-order B, whether there exists an optimal embedding of B into some order $\mathcal{O}'$ in the same spinor genus as $\mathcal{O}$ is encoded in the following *optimal spinor selectivity symbol*:

$$\Delta(B, \mathcal{O}) = \begin{cases} 1 & \text{if } \exists\, \mathcal{O}' \text{ such that } \mathcal{O}' \sim \mathcal{O} \text{ and } \mathrm{Emb}(B, \mathcal{O}') \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{1.9}$$

Clearly, if $\Delta(B, \mathcal{O}) = 1$, then $m_{\mathfrak{p}}(B) \neq 0$ for every finite prime $\mathfrak{p}$ of F. Conversely, whether the latter condition implies that $\Delta(B, \mathcal{O}) = 1$ or not is the central question of *the optimal spinor selectivity theory* [1,26,30,47] [41, §31]. A fundamental object of this theory is a class field Σ/F attached to the genus of $\mathcal{O}$ (see Definition 2.3). For example, if $m_{\mathfrak{p}}(B) \neq 0$ for every finite prime $\mathfrak{p}$ of F and the fractional field $\mathrm{Frac}(B)$ of B is not contained in Σ, then we do have $\Delta(B, \mathcal{O}) = 1$ by Theorem 2.5. Naturally, we define the following symbol

$$s(B, \mathcal{O}) = \begin{cases} 1 & \text{if } \mathrm{Frac}(B) \subseteq \Sigma; \\ 0 & \text{otherwise.} \end{cases} \tag{1.10}$$

A completely local characterization of the containment $\mathrm{Frac}(B) \subseteq \Sigma$ is given in Lemma 2.7. If $s(B, \mathcal{O}) = 1$, then $\Delta(B, \mathcal{O})$ can be computed using formulas (2.6), (2.7) and (2.8). Lastly, we write $h(F)$ (resp. $h^+(F)$) for the wide (resp. narrow) class number of F.

Different from the Eichler class number formula, our current class number formulas do not apply to any arbitrary $\mathcal{O}$. Rather, we need $\mathcal{O}$ to be *residually unramified*. See Definition 2.1 for the meaning of this notion, and Remark 2.9 for why such an assumption is necessary. Nevertheless, since any Eichler order is residually unramified by [41, Lemma 24.3.6], our formulas do apply to a large class of arithmetically important quaternion orders. Moreover, $\mathrm{Nr}(\widehat{\mathcal{O}}^\times) = \widehat{O}_F^\times$ for any residually unramified order $\mathcal{O}$ by (2.3), so the class number formula for $h_{\mathrm{sc}}(\mathcal{O})$ counts the neutral fiber of $\mathrm{Nr} : \mathrm{Cl}(\mathcal{O}) \to \mathrm{Cl}^+(O_F)$ as explained above.

Theorem 1.3. *Suppose that $\mathcal{O}$ is a residually unramified O_F-order in D. Then*

$$h^1(\mathcal{O}) = 2\,\mathrm{Mass}^1(\mathcal{O}) + \frac{1}{4h(F)} \sum_{B\in\mathscr{B}^1} 2^{s(B,\mathcal{O})}\Delta(B,\mathcal{O})(|\boldsymbol{\mu}(B)|-2)M(B), \tag{1.11}$$

$$h_{\mathrm{sc}}(\mathcal{O}) = \mathrm{Mass}_{\mathrm{sc}}(\mathcal{O}) + \frac{1}{2h^+(F)} \sum_{B\in\mathscr{B}} 2^{s(B,\mathcal{O})}\Delta(B,\mathcal{O})(w(B)-1)M(B). \tag{1.12}$$

Here $\mathrm{Mass}^1(\mathcal{O})$ *and* $\mathrm{Mass}_{\mathrm{sc}}(\mathcal{O})$ *can be computed by the mass formulas* (3.13) *and* (3.8) *respectively. The summation in (1.11) ranges over $\mathscr{B}^1$ while the one in (1.12) ranges over $\mathscr{B}$.*

If further D is ramified at some finite prime of F, then both $h^1(\mathcal{O})$ and $h_{\mathrm{sc}}(\mathcal{O})$ depend only on the genus of $\mathcal{O}$, and they are given by

$$h^1(\mathcal{O}) = 2\,\mathrm{Mass}^1(\mathcal{O}) + \frac{1}{4h(F)} \sum_{B\in\mathscr{B}^1} (|\boldsymbol{\mu}(B)|-2)M(B), \tag{1.13}$$

$$h_{\mathrm{sc}}(\mathcal{O}) = \mathrm{Mass}_{\mathrm{sc}}(\mathcal{O}) + \frac{1}{2h^+(F)} \sum_{B\in\mathscr{B}} (w(B)-1)M(B). \tag{1.14}$$

In this case we have $h_{\mathrm{sc}}(\mathcal{O}) = h(\mathcal{O})/h^+(F)$. Particularly, $h(\mathcal{O})$ is divisible by $h^+(F)$.

If we drop the condition that D is ramified at some finite prime of F, then $h(\mathcal{O})$ is not necessarily divisible by $h^+(F)$ anymore; see [44, Table 1, p. 676] for some examples. Nevertheless, as a direct application of (1.12), it has been shown by Yucui Lin and the first named author in [24] that $h(\mathcal{O})$ is always divisible by $h(F)$ regardless of the ramification of D over F. Such divisibility results closely mirror those of CM-extensions [42, Theorem 4.10]: namely, if K/F is a CM-extension, then $h(K)$ is divisible by $h(F)$, and if K/F is further assumed to be ramified at some finite prime of F, then $h(K)$ is divisible by $h^+(F)$.

Among the formulas in Theorem 1.3, the ones for $h^1(\mathcal{O})$ are most challenging to compute, so we devote the whole Section 4 to this task. Comparably, the formulas of $h_{\mathrm{sc}}(\mathcal{O})$ can be obtained by the same method for the Eichler class number formula, so they are merely sketched in the second half of Section 3.

Since D is totally definite, D^1 is discrete and cocompact in $\widehat{D}^1$. Hence a standard tool for computing $h^1(\mathcal{O})$ is the well-known Selberg trace formula for compact quotient. This formula consists of finitely many orbital integrals, each corresponding to a D^1-conjugacy class as in (4.15). Naively, one might expect that our formula for $h^1(\mathcal{O})$ is obtained simply by working out the orbital integrals one by one. This was the approach when we tried a few concrete examples, but immediately we are confronted with certain mysterious cancellations that cannot be explained by such a method; see Remark 4.4 for details. To uncover the mechanism behind this cancellation, we first cut the orbital integrals into pieces as in (4.26), with each piece giving rise to optimal embeddings of the same CM O_F-order B into $\mathcal{O}$. The pieces with the same B and corresponding to those D^1-conjugacy classes lying within the same $D^\times$-conjugacy class are then grouped together in (4.27). As we eventually discover in Proposition 4.5, the combined pieces of orbital integrals are related to the spinor trace formula developed by the authors in [47, Proposition 4.3] (see Proposition 2.8). However, such a connection is in no way obvious. Major efforts are spent in Section 4 to forge the links, and the main technicality of the present paper arises this way.

One of our motivations for developing the class number formulas for $h^1(\mathcal{O})$ and $h_{\mathrm{sc}}(\mathcal{O})$ is to count certain abelian surfaces over finite fields. From the Honda-Tate theorem [38, Theorem 1], given a prime number $p \in \mathbb{N}$, there is a unique isogeny class of simple abelian surfaces over the prime finite field $\mathbb{F}_p$ corresponding to the real Weil numbers $\pi = \pm\sqrt{p}$. Let $F = \mathbb{Q}(\sqrt{p})$, and $D_\pi := \mathrm{End}_{\mathbb{F}_p}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the endomorphism algebra of an arbitrary member $X/\mathbb{F}_p$ in this isogeny class. Then X is necessarily superspecial, and D_π is equal to the unique totally definite quaternion F-algebra D_{∞_1,∞_2} that is unramified at all finite places of F. In another paper [45] by the present authors, we work out explicitly the class numbers $h^1(\mathbb{O})$ and $h_{\mathrm{sc}}(\mathbb{O})$ for every maximal O_F-order $\mathbb{O}$ in D_{∞_1,∞_2}. It is shown there that $h^1(\mathbb{O})$ (resp. $h_{\mathrm{sc}}(\mathbb{O})$) counts the number of isomorphism classes of polarized (resp. unpolarized) abelian surfaces within certain genus in this isogeny class. Thus our class number formulas pave the way for explicit formulas for the number $|\mathrm{PPAV}(\sqrt{p})|$ of $\mathbb{F}_p$-isomorphism classes of principally polarized abelian surfaces in this isogeny class. For example, for $p > 5$ we show that $|\mathrm{PPAV}(\sqrt{p})|$ is equal to

$$\left(9 - 2\left(\frac{2}{p}\right)\right)\frac{\zeta_F(-1)}{2} + \frac{3h(-p)}{8} + \left(3 + \left(\frac{2}{p}\right)\right)\frac{h(-3p)}{6} \quad \text{if } p \equiv 1 \pmod 4,$$

and it is equal to

$$\frac{\zeta_F(-1)}{2} + \left(11 - 3\left(\frac{2}{p}\right)\right)\frac{h(-p)}{8} + \frac{h(-3p)}{6} \qquad \text{if } p \equiv 3 \pmod 4.$$

Here $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol, and $h(d)$ denotes the class number of the quadratic field $\mathbb{Q}(\sqrt{d})$.

This paper is organized as follows. In Section 2, we give a brief review of the optimal spinor selectivity theory. The meanings of the symbols $\Delta(B, \mathcal{O})$, $s(B, \mathcal{O})$ and their methods of calculation will be explained in more details. In Section 3, we (re)produce the mass formulas and sketch a proof for the formula of $h_{\mathrm{sc}}(\mathcal{O})$. Section 4 constitutes the core part of this paper, where we derive the formula for $h^1(\mathcal{O})$.

2. Optimal spinor selectivity theory

In this section, we review briefly the optimal spinor selectivity theory. Crucial to our class number formulas is the spinor trace formula (2.14), which involves both the symbols $\Delta(B, \mathcal{O})$ and $s(B, \mathcal{O})$. Our main reference to this section is the paper [47] by the present authors. See also Chapter 31 of Voight's book [41]. There is no need to assume that the quaternion algebra D is totally definite yet, so F is allowed to be an arbitrary number field in this section. Throughout this section, B denotes an O_F-order in a quadratic field extension K/F that embeds into D.

Given a locally principal right $\mathcal{O}$-ideal I in D, we write $[I]$ for its ideal class. The set of locally principal right $\mathcal{O}$-ideal classes is denoted by $\mathrm{Cl}(\mathcal{O})$. A key to the proof of the Eichler class number formula for $h(\mathcal{O}) := |\mathrm{Cl}(\mathcal{O})|$ is the following well-known *trace formula* [40, Theorem III.5.11] (cf. [41, Theorem 30.4.7], [44, Lemma 3.2.1] and [43, Lemma 3.2]) for optimal embeddings:

$$\sum_{[I]\in\mathrm{Cl}(\mathcal{O})} m(B, \mathcal{O}_l(I), \mathcal{O}_l(I)^\times) = h(B)\prod_{\mathfrak{p}} m_{\mathfrak{p}}(B), \tag{2.1}$$

where $\mathcal{O}_l(I)$ is the left order of I defined in (1.7), and the product runs over all finite primes $\mathfrak{p}$ of F. The spinor trace formula is a refinement of the above trace formula by grouping the right $\mathcal{O}$-ideal classes into spinor classes (see Definition 1.2). Unfortunately, for this refinement to hold without prior restrictions on B, we need to assume that $\mathcal{O}$ is *residually unramified*. To explain this notion, we recall the definition of *Eichler invariant* from [8, Definition 1.8] and [41, Definition 24.3.2].

Definition 2.1. (1) Let $\mathfrak{p}$ be a prime ideal of O_F, $\mathfrak{k}_\mathfrak{p} := O_F/\mathfrak{p}$ be the finite residue field, and $\mathfrak{k}'_\mathfrak{p}/\mathfrak{k}_\mathfrak{p}$ be the unique quadratic field extension. Let $O_{F_\mathfrak{p}}$ and $\mathcal{O}_\mathfrak{p}$ be the $\mathfrak{p}$-adic completions of O_F and $\mathcal{O}$ respectively. When $\mathcal{O}_\mathfrak{p}$ is not isomorphic to the matrix ring $M_2(O_{F_\mathfrak{p}})$, the quotient of $\mathcal{O}_\mathfrak{p}$ by its Jacobson radical $\mathfrak{J}(\mathcal{O}_\mathfrak{p})$ falls into the following three cases:

$$\mathcal{O}_\mathfrak{p}/\mathfrak{J}(\mathcal{O}_\mathfrak{p}) \simeq \mathfrak{k}_\mathfrak{p} \times \mathfrak{k}_\mathfrak{p}, \qquad \mathfrak{k}_\mathfrak{p}, \quad \text{or} \quad \mathfrak{k}'_\mathfrak{p},$$

and the *Eichler invariant* $e_\mathfrak{p}(\mathcal{O})$ of $\mathcal{O}$ at $\mathfrak{p}$ is defined to be $1, 0, -1$ accordingly. As a convention, if $\mathcal{O}_\mathfrak{p} \simeq M_2(O_{F_\mathfrak{p}})$, then its Eichler invariant at $\mathfrak{p}$ is defined to be 2.

(2) We say that $\mathcal{O}$ is *residually unramified at* $\mathfrak{p}$ if $e_\mathfrak{p}(\mathcal{O}) \neq 0$. If $e_\mathfrak{p}(\mathcal{O}) \neq 0$ for every finite prime $\mathfrak{p}$ of F, then we simply say that $\mathcal{O}$ is *residually unramified*.

For example, if $\mathfrak{p}$ is ramified in D and $\mathcal{O}_\mathfrak{p}$ is maximal, then $e_\mathfrak{p}(\mathcal{O}) = -1$. It is shown in [8, Proposition 2.1] that $e_\mathfrak{p}(\mathcal{O}) = 1$ if and only if $\mathcal{O}_\mathfrak{p}$ is a non-maximal Eichler order (particularly, $\mathfrak{p}$ is split in D). Let $\mathfrak{n}$ be a nonzero ideal of O_F such that no prime divisor of $\mathfrak{n}$ is ramified in D. If $\mathcal{O}$ is an Eichler order of level $\mathfrak{n}$, then

$$e_\mathfrak{p}(\mathcal{O}) = \begin{cases} 1 & \text{if } \mathfrak{p}|\mathfrak{n}; \\ -1 & \text{if } \mathfrak{p} \text{ is ramified in } D; \\ 2 & \text{otherwise.} \end{cases} \tag{2.2}$$

This shows that all Eichler orders are residually unramified.

Remark 2.2. From [8, Corollary 2.4 and Proposition 3.1], a residually unramified order $\mathcal{O}$ is automatically *Bass* (in particular, *Gorenstein*), so any O_F-lattice L of full rank in D with $\mathcal{O}_l(L) = \mathcal{O}$ is automatically locally principal as a left $\mathcal{O}$-ideal by [10, Example 2.6]. Moreover, it has been shown in the proof of [47, Lemma 2.17] that if $\mathcal{O}$ is residually unramified, then

$$\mathrm{Nr}(\widehat{\mathcal{O}}^\times) = \widehat{O}_F^\times. \tag{2.3}$$

We return to the assumption that $\mathcal{O}$ is arbitrary. By definition, a genus of orders in D is an equivalence class of orders that are locally isomorphic at every finite prime of F. The notion of *spinor genus of O_F-orders* has already appeared in Definition 1.2. Let $\mathscr{G}$ (resp. $[\mathcal{O}]_{\mathrm{sg}}$) be the genus (resp. spinor genus) of $\mathcal{O}$. More explicitly,

$$\mathscr{G} := \{\mathcal{O}' \mid \exists x \in \widehat{D}^\times \text{ such that } \widehat{\mathcal{O}}' = x\widehat{\mathcal{O}}x^{-1}\},$$
$$[\mathcal{O}]_{\mathrm{sg}} := \{\mathcal{O}' \mid \exists x \in D^\times \widehat{D}^1 \text{ such that } \widehat{\mathcal{O}}' = x\widehat{\mathcal{O}}x^{-1}\} \subseteq \mathscr{G}.$$

The set of spinor genera within $\mathscr{G}$ is denoted by $\mathrm{SG}(\mathscr{G})$, that is, $\mathrm{SG}(\mathscr{G}) := \{[\mathcal{O}']_{\mathrm{sg}} \mid \mathcal{O}' \in \mathscr{G}\}$. Often we write $\mathrm{SG}(\mathcal{O})$ for $\mathrm{SG}(\mathscr{G})$ and regard it as a pointed set with the base point $[\mathcal{O}]_{\mathrm{sg}}$. For simplicity let us put $F_D^\times = \mathrm{Nr}(D^\times)$, where $\mathrm{Nr} : D^\times \to F^\times$ is the reduced norm map. From the Hasse-Schilling-Maass theorem [35, Theorem 33.15] [40, Theorem III.4.1], $F_D^\times$ coincides with the subgroup of $F^\times$ consisting of the elements that are positive at each infinite place of F ramified in D. Let $\mathcal{N}(\widehat{\mathcal{O}})$ be the normalizer of $\widehat{\mathcal{O}}$ in $\widehat{D}^\times$. The pointed set $\mathrm{SG}(\mathcal{O})$ can be described adelically as follows

$$\mathrm{SG}(\mathcal{O}) \simeq (D^\times \widehat{D}^1)\backslash \widehat{D}^\times/\mathcal{N}(\widehat{\mathcal{O}}) \xrightarrow[\simeq]{\mathrm{Nr}} F_D^\times\backslash \widehat{F}^\times/\,\mathrm{Nr}(\mathcal{N}(\widehat{\mathcal{O}})), \tag{2.4}$$

where the two double coset spaces are canonically bijective via the reduced norm map. Clearly, $\mathrm{Nr}(\mathcal{N}(\widehat{\mathcal{O}}))$ depends only on the genus $\mathscr{G}$ and not on the particular choice of $\mathcal{O} \in \mathscr{G}$.

Definition 2.3 ([2, §2], [25, §3])**.** The *spinor genus field*[1] of $\mathscr{G}$ is the abelian field extension Σ/F corresponding to the open subgroup $F_D^\times \mathrm{Nr}(\mathcal{N}(\widehat{\mathcal{O}})) \subseteq \widehat{F}^\times$ via the class field theory [22, Theorem X.5].

Since $\mathrm{Nr}(\mathcal{N}(\widehat{\mathcal{O}}))$ is an open subgroup of $\widehat{F}^\times$ containing $(\widehat{F}^\times)^2$, the Galois group $\mathrm{Gal}(\Sigma/F)$ is a finite elementary 2-group [25, Proposition 3.5]. We have a canonical identification of pointed sets

$$\mathrm{SG}(\mathcal{O}) \simeq F_D^\times\backslash \widehat{F}^\times/\,\mathrm{Nr}(\mathcal{N}(\widehat{\mathcal{O}})) \simeq \mathrm{Gal}(\Sigma/F), \tag{2.5}$$

where the base point $[\mathcal{O}]_{\mathrm{sg}}$ is identified with the identity element of $\mathrm{Gal}(\Sigma/F)$. Given another order $\mathcal{O}' \in \mathscr{G}$, we define $\rho(\mathcal{O},\mathcal{O}')$ to be the element of $\mathrm{Gal}(\Sigma/F)$ identified with $[\mathcal{O}']_{\mathrm{sg}} \in \mathrm{SG}(\mathcal{O})$ via (2.5). More explicitly, if $\widehat{\mathcal{O}}' = x\widehat{\mathcal{O}}x^{-1}$ for some $x \in \widehat{D}^\times$, then $\rho(\mathcal{O},\mathcal{O}') = (\mathrm{Nr}(x), \Sigma/F)$, where $a \mapsto (a, \Sigma/F)$ is the Artin map on the finite idele group $\widehat{F}^\times$. Write the group law of $\mathrm{Gal}(\Sigma/F)$ additively. Then $\rho(\mathcal{O},\mathcal{O}')$ enjoys the following properties:

(a) $\rho(\mathcal{O},\mathcal{O}') = 0$ if and only if $\mathcal{O} \sim \mathcal{O}'$;
(b) $\rho(\mathcal{O},\mathcal{O}') = \rho(\mathcal{O}',\mathcal{O})$;
(c) $\rho(\mathcal{O},\mathcal{O}'') = \rho(\mathcal{O},\mathcal{O}') + \rho(\mathcal{O}',\mathcal{O}'')$.

When $\mathscr{G}$ is a genus of residually unramified orders, $\rho(\mathcal{O},\mathcal{O}')$ can be computed as follows. There exists an O_F-lattice $I \subset D$ *linking* $\mathcal{O}$ and $\mathcal{O}'$ in the sense that I is locally principal right $\mathcal{O}$-ideal with $\mathcal{O}_l(I) = \mathcal{O}'$. Then $\rho(\mathcal{O},\mathcal{O}') \in \mathrm{Gal}(\Sigma/F)$ is given by the Artin symbol $(\mathrm{Nr}(I), \Sigma/F)$. Here the Artin symbol is well-defined since $\mathrm{Nr}(\widehat{\mathcal{O}}^\times) = \widehat{O}_F^\times$ by (2.3), so Σ/F is unramified at all the finite places of F.

Now let $\mathscr{G}$ be an arbitrary genus of O_F-orders in D. Recall that B denotes an O_F-order in a quadratic field extension K/F that embeds into D. Let $\Delta(B,\mathcal{O})$ be the symbol defined in (1.9). The reason that it is called the *optimal spinor selectivity symbol* is as follows.

Definition 2.4. We say B is *optimally spinor selective* for the genus $\mathscr{G}$ if $\Delta(B,\mathcal{O}) = 1$ for some but not all $[\mathcal{O}]_{\mathrm{sg}} \in \mathrm{SG}(\mathscr{G})$. If B is selective for $\mathscr{G}$, then a spinor genus $[\mathcal{O}]_{\mathrm{sg}}$ with $\Delta(B,\mathcal{O}) = 1$ is said to be *selected* by B.

As mentioned in the introduction, if $\Delta(B,\mathcal{O}) = 1$ for some $\mathcal{O} \in \mathscr{G}$, then $m_{\mathfrak{p}}(B) \neq 0$ for every finite prime $\mathfrak{p}$ of F. Note that the latter condition depends only on $\mathscr{G}$ and not on the particular choice of $\mathcal{O} \in \mathscr{G}$. Conversely, whether and when the condition $m_{\mathfrak{p}}(B) \neq 0$ for every $\mathfrak{p}$ implies that $\Delta(B,\mathcal{O}) = 1$ for every $[\mathcal{O}] \in \mathrm{SG}(\mathscr{G})$ are the central questions of the optimal spinor selectivity theory. The importance of the symbol $s(B,\mathcal{O})$ defined in (1.10)

[1] This field is often called the *spinor class field* in the literature [2, 3, 4], but that can be easily mixed up with the notion of *spinor class* here. On the other hand, if F is a quadratic field, and $\mathcal{O}$ is an Eichler order, then $\Sigma_{\mathscr{G}}$ is a subfield of the classical *(strict) genus field* in [13, Definition 15.29] or [15, §6], so the terminology is consistent in that sense.

is clear by the following theorem, which is obtained by combining [47, Theorem 2.15 and Corollary 2.16].

Theorem 2.5. *Let $\mathscr{G}$ be a genus of residually unramified O_F-orders in D, and B be an O_F-order in a quadratic field extension K/F that embeds into D. Suppose that $m_{\mathfrak{p}}(B) \neq 0$ for every finite prime $\mathfrak{p}$ of F. Then B is optimally spinor selective for $\mathscr{G}$ if and only if $K \subseteq \Sigma$. If B is optimally spinor selective, then*

(1) for any two orders $\mathcal{O}, \mathcal{O}' \in \mathscr{G}$,

$$\Delta(B, \mathcal{O}) = \rho(\mathcal{O}, \mathcal{O}')|_K + \Delta(B, \mathcal{O}'), \tag{2.6}$$

where $\rho(\mathcal{O}, \mathcal{O}')|_K$ is the restriction of $\rho(\mathcal{O}, \mathcal{O}') \in \mathrm{Gal}(\Sigma/F)$ to K, and the summation on the right is taken inside $\mathbb{Z}/2\mathbb{Z}$ with the canonical identification $\mathrm{Gal}(K/F) \simeq \mathbb{Z}/2\mathbb{Z}$;

(2) exactly half of the spinor genera in $\mathrm{SG}(\mathscr{G})$ are selected by B.

The above theorem was first obtained by Maclachlan [26] for Eichler orders of square-free levels, and it was extended to Eichler orders of arbitrary levels independently by Arenas et al [1] and by Voight [41, Chapter 31]. See [47, Theorem 2.15] and [30, Theorem 2.6] for more general theorems where the "residually unramified" assumption is dropped.

As soon as the value of $\Delta(B, \mathcal{O})$ is known for one order $\mathcal{O}$, we can use formula (2.6) to compute all other $\Delta(B, \mathcal{O}')$. This formula can be generalized a little further when $\mathscr{G}$ is a genus of Eichler orders. Let $\mathfrak{f}(B)$ be the conductor of B, that is, $\mathfrak{f}(B)$ is the unique O_F-ideal $\mathfrak{f}(B) \subseteq O_F$ such that $B = O_F + \mathfrak{f}(B)O_K$. If B' is another order in K, we put $\mathfrak{f}(B'/B) := \mathfrak{f}(B')^{-1}\mathfrak{f}(B)$ and call it the *relative conductor* of B with respect to B'.

Proposition 2.6. *Let $\mathscr{G}$ be a genus of Eichler orders in D, and K be a quadratic field extension of F that embeds into D. Let B and B' be O_F-orders in K satisfying $m_{\mathfrak{p}}(B) \neq 0$ and $m_{\mathfrak{p}}(B') \neq 0$ for all finite primes $\mathfrak{p}$ of F. Suppose that $K \subseteq \Sigma$ so that both B and B' are optimally spinor selective for $\mathscr{G}$. Then*

$$\Delta(B, \mathcal{O}) = \big(\mathfrak{f}(B'/B), K/F\big) + \rho(\mathcal{O}, \mathcal{O}')|_K + \Delta(B', \mathcal{O}'), \tag{2.7}$$

where $\big(\mathfrak{f}(B'/B), K/F\big)$ is the Artin symbol. In particular, if $B' = \varphi^{-1}(\mathcal{O})$ for some F-embedding $\varphi : K \to D$, then

$$\Delta(B, \mathcal{O}) = \big(\mathfrak{f}(B'/B), K/F\big) + 1. \tag{2.8}$$

Here the assumption $K \subseteq \Sigma$ implies that K is unramified at all the finite places of F by Lemma 2.7 below, so the Artin symbol $\big(\mathfrak{f}(B'/B), K/F\big)$ is well-defined. The proposition above is first obtained by Maclachlan [26] for Eichler orders of square-free levels, and it is extended to Eichler orders of arbitrary levels by the current authors in [47, §3].

We give a local criterion for the inclusion $K \subseteq \Sigma$, which enables us to compute the symbol $s(B, \mathcal{O})$ easily when $\mathcal{O}$ is residually unramfied (cf. [30, Proposition 2.9]). Let $\mathfrak{d}(\mathcal{O})$ be the *reduced discriminant* [40, §I.4, p. 24] of $\mathcal{O}$, which is an integral ideal of O_F. For each prime $\mathfrak{p}$, we write $\nu_{\mathfrak{p}} : F^\times \twoheadrightarrow \mathbb{Z}$ for the normalized discrete valuation of F attached to $\mathfrak{p}$.

Lemma 2.7. *Let $\mathscr{G}$ be a genus of residually unramified orders in D, and K be a quadratic field extension of F that embeds into D. Then $K \subseteq \Sigma$ if and only if both of the following conditions hold:*

(a) the extension K/F and the quaternion F-algebra D are unramified at every finite prime $\mathfrak{p}$ of F and ramify at exactly the same (possibly empty) set of infinite places;

(b) if $\mathfrak{p}$ is a finite prime of F with $\nu_{\mathfrak{p}}(\mathfrak{d}(\mathcal{O})) \equiv 1 \pmod 2$, then $\mathfrak{p}$ splits in K.

Once again the above lemma is a generalization of [41, Proposition 31.2.1] (see also [1, Theorem 1.1(3)] and [25, Proposition 5.11]). It is obtained in the current form by the authors in [47, Lemma 2.17].

Finally, we move on to the spinor trace formula. For the moment, we assume that $\mathcal{O}$ is an arbitrary O_F-order in D. The notion of *spinor class* of locally principal right $\mathcal{O}$-ideals has already appeared in Definition 1.2. Note that if I and I' belong to the same spinor class, then their left orders $\mathcal{O}_l(I)$ and $\mathcal{O}_l(I')$ belong to the same spinor genus. The set of all locally principal right $\mathcal{O}$-ideals I' in the same spinor class as I will be denoted by $[I]_{\mathrm{sc}}$. Let $\mathrm{Cl}(\mathcal{O}, [I]_{\mathrm{sc}})$ be the set of ideal classes in $[I]_{\mathrm{sc}}$, that is

$$\mathrm{Cl}(\mathcal{O}, [I]_{\mathrm{sc}}) := \{[I'] \in \mathrm{Cl}(\mathcal{O}) \mid [I'] \subseteq [I]_{\mathrm{sc}}\}. \tag{2.9}$$

With this notation, the set $\mathrm{Cl}_{\mathrm{sc}}(\mathcal{O})$ considered in the introduction is just $\mathrm{Cl}(\mathcal{O}, [\mathcal{O}]_{\mathrm{sc}})$.

Let $\mathrm{SCl}(\mathcal{O})$ be the set of spinor classes of locally principal right $\mathcal{O}$-ideals, regarded as a pointed set with base point $[\mathcal{O}]_{\mathrm{sc}}$. It admits an adelic description as follows

$$\mathrm{SCl}(\mathcal{O}) \simeq (D^\times \widehat{D}^1)\backslash \widehat{D}^\times / \widehat{\mathcal{O}}^\times \underset{\simeq}{\xrightarrow{\mathrm{Nr}}} F_D^\times \backslash \widehat{F}^\times / \operatorname{Nr}(\widehat{\mathcal{O}}^\times), \tag{2.10}$$

where the two double coset spaces are canonically bijective via the reduced norm map. This equips $\mathrm{SCl}(\mathcal{O})$ with a group structure (whose identity element is $[\mathcal{O}]_{\mathrm{sc}}$), so we call it the *spinor class group*.

By definition, the ideal class set $\mathrm{Cl}(\mathcal{O})$ is partitioned into a finite disjoint union

$$\mathrm{Cl}(\mathcal{O}) = \coprod_{[I]_{\mathrm{sc}} \in \mathrm{SCl}(\mathcal{O})} \mathrm{Cl}(\mathcal{O}, [I]_{\mathrm{sc}}). \tag{2.11}$$

Let I^{-1} be the inverse ideal of I as defined in [40, §I.4, p. 21], that is, $I^{-1} := \{\alpha \in D \mid I\alpha I \subseteq I\}$. Then I^{-1} is a locally principal left $\mathcal{O}$-ideal whose right order coincides with $\mathcal{O}_l(I) = II^{-1}$. Right multiplication by I^{-1} induces a bijection

$$\{\text{locally principal right } \mathcal{O}\text{-ideals}\} \to \{\text{locally principal right } \mathcal{O}_l(I)\text{-ideals}\},$$

which preserves spinor classes, ideal classes, and left orders. In particular, it induces a bijection

$$\mathrm{Cl}(\mathcal{O}, [I]_{\mathrm{sc}}) \simeq \mathrm{Cl}_{\mathrm{sc}}(\mathcal{O}_l(I)). \tag{2.12}$$

From (2.11) we get

$$h(\mathcal{O}) = \sum_{[I]_{\mathrm{sc}} \in \mathrm{SCl}(\mathcal{O})} h_{\mathrm{sc}}(\mathcal{O}_l(I)), \tag{2.13}$$

so the class number formula for $h_{\mathrm{sc}}(\mathcal{O})$ is indeed a refinement of the Eichler class number formula for $h(\mathcal{O})$. As mentioned in the introduction, $h_{\mathrm{sc}}(\mathcal{O})$ depends only on the spinor genus of $\mathcal{O}$, so $h_{\mathrm{sc}}(\mathcal{O}_l(I))$ does not depend on the choice of the representative I of $[I]_{\mathrm{sc}}$.

Following [40, §III.5, p. 88], we define the *restricted class number of F with respect to D* as $h_D(F) := |\widehat{F}^\times / (F_D^\times \widehat{O}_F^\times)|$. If $\mathcal{O}$ is residually unramified, then $\operatorname{Nr}(\widehat{\mathcal{O}}^\times) = \widehat{O}_F^\times$ by (2.3). Thus $|\mathrm{SCl}(\mathcal{O})| = h_D(F)$ for residually unramified $\mathcal{O}$ by (2.10). Now the following spinor trace formula is a special case of the one given in [47, Proposition 4.3].

Proposition 2.8 (Spinor trace formula)**.** *Let $\mathcal{O}$ be a residually unramified O_F-order in D, and B be an O_F-order in a quadratic field extension K/F that embeds into D. Then we have*

$$\sum_{[I]\in\mathrm{Cl}_{\mathrm{sc}}(\mathcal{O})} m(B,\mathcal{O}_l(I),\mathcal{O}_l(I)^\times) = \frac{2^{s(B,\mathcal{O})}\Delta(B,\mathcal{O})h(B)}{h_D(F)}\prod_{\mathfrak{p}} m_{\mathfrak{p}}(B). \tag{2.14}$$

If further D is assumed to be ramified at some finite prime of F, then

$$\sum_{[I]\in\mathrm{Cl}_{\mathrm{sc}}(\mathcal{O})} m(B,\mathcal{O}_l(I),\mathcal{O}_l(I)^\times) = \frac{h(B)}{h_D(F)}\prod_{\mathfrak{p}} m_{\mathfrak{p}}(B). \tag{2.15}$$

Note that (2.15) is just a specialized form of (2.14). Indeed, if D is ramified at some finite place of F, then $s(B,\mathcal{O})=0$ (that is, $\mathrm{Frac}(B)\not\subseteq\Sigma$) by Lemma 2.7. Now it follows from Theorem 2.5 that $\Delta(B,\mathcal{O})\prod_{\mathfrak{p}} m_{\mathfrak{p}}(B)=\prod_{\mathfrak{p}} m_{\mathfrak{p}}(B)$. More explicitly, if $m_{\mathfrak{p}}(B)\neq 0$ for every finite prime $\mathfrak{p}$ of F, then $\Delta(B,\mathcal{O})=1$, otherwise $\Delta(B,\mathcal{O})=\prod_{\mathfrak{p}} m_{\mathfrak{p}}(B)=0$.

The spinor trace formula is a refinement of the trace formula (2.1). When D satisfies the Eichler condition (that is, D is indefinite), Brzezinski [9, Proposition 1.1] shows that each spinor genus of O_F-orders contains exactly one $D^\times$-conjugacy class, and each spinor class of locally principal right $\mathcal{O}$-ideals contains exactly one ideal class. Thus in this case $\mathrm{Cl}_{\mathrm{sc}}(\mathcal{O})$ is a singleton with the unique member $[\mathcal{O}]$, and the summation on the left hand of (2.14) contains only the term $m(B,\mathcal{O},\mathcal{O}^\times)$.

The spinor trace formula was first studied by Vignéras in the indefinite context for Eichler orders in [40, Corollaire III.5.17]. However, it was pointed out by Chinburg and Friedman in [12, Remark 3.4] that her formula needs to be adjusted to account for the overlooked exceptional cases where (optimal spinor) selectivity does occur. Indeed, it was them who coined the term “selectivity”. The corrected formula was worked out by Voight in [41, Theorem 31.1.7(c) and Corollary 31.1.10] under the same assumption as Vignéras's. The formula is further generalized to the totally definite case by the current authors for the purpose of the present paper.

Remark 2.9. The reason that we assume $\mathcal{O}$ to be residually unramified is that only in this case, the spinor trace formula (2.14) is currently known to hold without further restrictions on B. This is also precisely the reason why we make the same assumption on $\mathcal{O}$ in Theorem 1.3. As soon as this assumption is dropped, the criterion for optimal spinor selectivity can become very complicated. Compare Theorem 2.5 with [30, Theorem 2.6]. In the more general case, there are examples of pair of orders $\mathcal{O}$ and B in [30, §5] where $s(B,\mathcal{O})=1$ yet B is non-selective for the genus of $\mathcal{O}$. For such orders it is unknown whether (2.14) remains true or not. See [47, Remark 4.4] for more details.

Lastly, suppose that F is a totally real field, and D is a totally definite quaternion algebra. We provide a formula for $\mathrm{SCl}(\mathcal{O})$ in this case for an arbitrary $\mathcal{O}$. Let $F_+^\times$ be the group of totally positive elements of $F^\times$, and $O_{F,+}^\times := F_+^\times\cap O_F^\times$. The subgroup of $O_{F,+}^\times$ consisting of all perfect squares in $O_F^\times$ is denoted by $O_F^{\times 2}$. Since D is totally definite, $F_D^\times=F_+^\times$, and $h_D(F)=h^+(F)$, the narrow class number of F. From [14, Lemma 11.6], we have

$$h^+(F)=h(F)[O_{F,+}^\times : O_F^{\times 2}]. \tag{2.16}$$

Lemma 2.10. *Let F and D be as above, and $\mathcal{O}$ be an arbitrary order in D. Then*

$$|\mathrm{SCl}(\mathcal{O})| = h(F)[\widehat{O}_F^\times : \mathrm{Nr}(\widehat{\mathcal{O}}^\times)][(O_{F,+}^\times\cap\mathrm{Nr}(\widehat{\mathcal{O}}^\times)) : O_F^{\times 2}]. \tag{2.17}$$

In particular, if $\mathcal{O}$ is residually unramified, then $|\mathrm{SCl}(\mathcal{O})| = h^+(F)$.

Proof. A straightforward calculation shows that

$$\frac{|\mathrm{SCl}(\mathcal{O})|}{h^+(F)} = [F_+^\times \widehat{O}_F^\times : F_+^\times \mathrm{Nr}(\widehat{\mathcal{O}}^\times)] = \frac{[\widehat{O}_F^\times : \mathrm{Nr}(\widehat{\mathcal{O}}^\times)]}{[O_{F,+}^\times : (O_{F,+}^\times \cap \mathrm{Nr}(\widehat{\mathcal{O}}^\times))]}. \tag{2.18}$$

Plugging (2.16) into (2.18), we obtain (2.17) by observing that $(O_{F,+}^\times \cap \mathrm{Nr}(\widehat{\mathcal{O}}^\times)) \supseteq O_F^{\times 2}$. The last part is a special case of the identity $|\mathrm{SCl}(\mathcal{O})| = h_D(F)$ for residually unramified orders as observed right before Proposition 2.8. $\square$

3. The Mass formulas and the formula for $h_{\mathrm{sc}}(\mathcal{O})$

The goal of this section is twofold. We first (re)produce the mass formulas for $\mathrm{Mass}^1(\mathcal{O})$ and $\mathrm{Mass}_{\mathrm{sc}}(\mathcal{O})$ following the expositions in [44, §5] and [48, §2], and then move on to the class number formula for $h_{\mathrm{sc}}(\mathcal{O}) := |\mathrm{Cl}_{\mathrm{sc}}(\mathcal{O})|$. The derivation of the formula for $h_{\mathrm{sc}}(\mathcal{O})$ follows exactly the same line of argument as that for the Eichler class number formula [40, Corollaire V.2.5, p. 144], so we merely provide a brief sketch. Throughout this section, F denotes a totally real field, and D a totally definite quaternion F-algebra. For each nonzero ideal $\mathfrak{a} \subseteq O_F$, we put $\mathrm{N}(\mathfrak{a}) = |O_F/\mathfrak{a}|$.

Let $\mathcal{O}$ be an arbitrary O_F-order in D with reduced discriminant $\mathfrak{d}(\mathcal{O})$, and $\mathrm{Cl}(\mathcal{O})$ be the right $\mathcal{O}$-ideal class set. Since D is totally definite, $\mathcal{O}^\times/O_F^\times$ is a finite group by [40, Theorem V.1.2]. By definition, the *mass* of $\mathcal{O}$ is the following weighted sum

$$\mathrm{Mass}(\mathcal{O}) := \sum_{[I]\in \mathrm{Cl}(\mathcal{O})} \frac{1}{[\mathcal{O}_l(I)^\times : O_F^\times]}, \tag{3.1}$$

where $\mathcal{O}_l(I)$ is the left order of I in (1.7). It is first shown by Körner [21] (cf. [48, Corollary 4.3]) that

$$\mathrm{Mass}(\mathcal{O}) = \frac{h(F)|\zeta_F(-1)|\,\mathrm{N}(\mathfrak{d}(\mathcal{O}))}{2^{n-1}} \prod_{\mathfrak{p}|\mathfrak{d}(\mathcal{O})} \frac{1-\mathrm{N}(\mathfrak{p})^{-2}}{1-e_\mathfrak{p}(\mathcal{O})\,\mathrm{N}(\mathfrak{p})^{-1}}. \tag{3.2}$$

where $n = [F : \mathbb{Q}]$, and $e_\mathfrak{p}(\mathcal{O})$ is the Eichler invariant in Definition 2.1. Here $\zeta_F(s)$ is the Dedekind ζ-function of F, whose special values at all negative odd integers are rational [50, Theorem, p. 59]. It is easy to see from the functional equation [22, §XIII.3, Theorem 2] that $\mathrm{sgn}(\zeta_F(-1)) = (-1)^n$.

As usual, the spinor class mass is a refinement of $\mathrm{Mass}(\mathcal{O})$. For each spinor class $[J]_{\mathrm{sc}} \in \mathrm{SCl}(\mathcal{O})$, we define

$$\mathrm{Mass}(\mathcal{O}, [J]_{\mathrm{sc}}) := \sum_{[I]\in \mathrm{Cl}(\mathcal{O},[J]_{\mathrm{sc}})} \frac{1}{[\mathcal{O}_l(I)^\times : O_F^\times]}, \tag{3.3}$$

where $\mathrm{Cl}(\mathcal{O}, [J]_{\mathrm{sc}})$ denotes the set of ideal classes in $[J]_{\mathrm{sc}}$ as in (2.9). For simplicity, put $\mathrm{Mass}_{\mathrm{sc}}(\mathcal{O}) := \mathrm{Mass}(\mathcal{O}, [\mathcal{O}]_{\mathrm{sc}})$.

Lemma 3.1. *The mass is equi-distributed among the spinor classes, so*

$$\mathrm{Mass}(\mathcal{O}, [J]_{\mathrm{sc}}) = \frac{\mathrm{Mass}(\mathcal{O})}{|\mathrm{SCl}(\mathcal{O})|}, \qquad \forall\ [J]_{\mathrm{sc}} \in \mathrm{SCl}(\mathcal{O}). \tag{3.4}$$

Proof. We recall the volume interpretation of $\mathrm{Mass}(\mathcal{O})$ from [44, §5]. Consider the following groups

$$H := \widehat{D}^\times/\widehat{O}_F^\times, \qquad U := \widehat{\mathcal{O}}^\times/\widehat{O}_F^\times, \qquad \Omega := D^\times\widehat{O}_F^\times/\widehat{O}_F^\times \simeq D^\times/O_F^\times. \tag{3.5}$$

Clearly, H is a locally compact topological group, and U is an open compact subgroup of H. It is well known that the group of finite adelic points of a connected reductive linear algebraic group over F is unimodular. In particular, $\widehat{D}^\times$ is unimodular, and hence H itself is unimodular by [28, Proposition 22, §II.5]. We normalize the Haar measure on H so that $\mathrm{Vol}(U) = 1$. Since $\mathcal{O}^\times/O_F^\times$ is finite, the group Ω is *discrete* in H, and it is also cocompact by [40, Theoreme Fondamental, pp. 61–62] or [33, Theorem 5.2]. Equip Ω with the counting measure. From [44, Lemma 5.1.1], we have $\mathrm{Mass}(\mathcal{O}) = \mathrm{Vol}(\Omega\backslash H)$, where the homogeneous space $\Omega\backslash H$ is equipped with the unique induced right H-invariant measure [28, Corollary 4, §III.4]. Given a locally principal right $\mathcal{O}$-ideal J, we write $\widehat{J} = x\widehat{\mathcal{O}}$ for some $x \in \widehat{D}^\times$. From (2.10), the spinor class $[J]_{\mathrm{sc}} \in \mathrm{SCl}(\mathcal{O})$ corresponds to the double coset $D^\times\widehat{D}^1x\widehat{\mathcal{O}}^\times$. The same proof as that of [44, Lemma 5.1.1] shows that

$$\begin{aligned}\mathrm{Mass}(\mathcal{O}, [J]_{\mathrm{sc}}) &= \mathrm{Vol}\left(\Omega\backslash(D^\times\widehat{D}^1x\widehat{\mathcal{O}}^\times/\widehat{O}_F^\times)\right) \overset{(\dagger)}{=\!=} \mathrm{Vol}\left(\Omega\backslash(D^\times\widehat{D}^1x\widehat{\mathcal{O}}^\times x^{-1}/\widehat{O}_F^\times)\right)\\ &\overset{(\ddagger)}{=\!=} \mathrm{Vol}\left(\Omega\backslash(D^\times\widehat{D}^1\widehat{\mathcal{O}}^\times/\widehat{O}_F^\times)\right) = \mathrm{Mass}(\mathcal{O}, [\mathcal{O}]_{\mathrm{sc}}).\end{aligned}$$

Here $(\dagger)$ follows from the right H-invariance of the measure on $\Omega\backslash H$, and $(\ddagger)$ follows from the equality $\widehat{D}^1x\widehat{\mathcal{O}}^\times x^{-1} = \widehat{D}^1\widehat{\mathcal{O}}^\times$. The lemma is proved. □

A formula for $|\mathrm{SCl}(\mathcal{O})|$ has already been worked out in (2.17). Let us put

$$u(\mathcal{O}) := [(O_{F,+}^\times \cap \mathrm{Nr}(\widehat{\mathcal{O}}^\times)) : O_F^{\times 2}]. \tag{3.6}$$

Combining (3.4) and (2.17), we get

$$\mathrm{Mass}_{\mathrm{sc}}(\mathcal{O}) = \frac{|\zeta_F(-1)|\,\mathrm{N}(\mathfrak{d}(\mathcal{O}))}{2^{n-1}[\widehat{O}_F^\times : \mathrm{Nr}(\widehat{\mathcal{O}}^\times)]u(\mathcal{O})}\prod_{\mathfrak{p}|\mathfrak{d}(\mathcal{O})}\frac{1-\mathrm{N}(\mathfrak{p})^{-2}}{1-e_{\mathfrak{p}}(\mathcal{O})\,\mathrm{N}(\mathfrak{p})^{-1}}. \tag{3.7}$$

For example, if $\mathcal{O}$ is residually unramified, then $\mathrm{Nr}(\widehat{\mathcal{O}}^\times) = \widehat{O}_F^\times$ by (2.3), and

$$\mathrm{Mass}_{\mathrm{sc}}(\mathcal{O}) = \frac{|\zeta_F(-1)|\,\mathrm{N}(\mathfrak{d}(\mathcal{O}))}{2^{n-1}[O_{F,+}^\times : O_F^{\times 2}]}\prod_{\mathfrak{p}|\mathfrak{d}(\mathcal{O})}\frac{1-\mathrm{N}(\mathfrak{p})^{-2}}{1-e_{\mathfrak{p}}(\mathcal{O})\,\mathrm{N}(\mathfrak{p})^{-1}}. \tag{3.8}$$

Next, we recall the definition and meaning of $\mathrm{Mass}^1(\mathcal{O})$. Let $\underline{D}^1$ be reduced norm one group of D, regarded as semisimple algebraic group over F. Functorially, $\underline{D}^1$ represents the functor sending every commutative F-algebra R to

$$\underline{D}^1(R) := \{g \in (D \otimes_F R)^\times \mid \mathrm{Nr}(g) = \bar{g}g = 1\}. \tag{3.9}$$

Thus $\widehat{D}^1 = \underline{D}^1(\widehat{F})$ is a locally compact unimodular group, and $\widehat{\mathcal{O}}^1$ is an open compact subgroup of $\widehat{D}^1$. We normalize the Haar measure on $\widehat{D}^1$ so that $\mathrm{Vol}(\widehat{\mathcal{O}}^1) = 1$. From [40, Lemma V.1.1], $\mathcal{O}^1 = D^1 \cap \widehat{\mathcal{O}}^1$ is a finite group, so $D^1 = \underline{D}^1(F)$ is discrete in $\widehat{D}^1$, and we equip it with the counting measure. From the equivalent conditions in [48, Proposition 2.1], D^1 is cocompact in $\widehat{D}^1$, which is also clear by [33, Theorem 5.2]. For simplicity, put $\ell := h^1(\mathcal{O})$. Let $x_1, \cdots, x_\ell \in \widehat{D}^1$ be a complete set of representatives for the double coset space $D^1\backslash\widehat{D}^1/\widehat{\mathcal{O}}^1$.

For each $1 \leq i \leq \ell$, we write $\mathcal{O}_i$ for the O_F-order $D \cap x_i \widehat{\mathcal{O}} x_i^{-1}$. By definition, $\mathrm{Mass}^1(\mathcal{O})$ is the following weighted sum

$$\mathrm{Mass}^1(\mathcal{O}) := \sum_{i=1}^{\ell} |D^1 \cap x_i \widehat{\mathcal{O}}^1 x_i^{-1}|^{-1} = \sum_{i=1}^{\ell} |\mathcal{O}_i^1|^{-1}. \tag{3.10}$$

Thanks to [48, Lemma 2.2], we have

$$\mathrm{Mass}^1(\mathcal{O}) = \mathrm{Vol}(D^1 \backslash \widehat{D}^1). \tag{3.11}$$

It follows that $\mathrm{Mass}^1(\mathcal{O})$ depends only on the genus of $\mathcal{O}$.

Combining [48, Theorem 3.2, Corollary 3.8 and Corollary 4.3], we obtain

$$\mathrm{Mass}^1(\mathcal{O}) = \frac{|\zeta_F(-1)| \, \mathrm{N}(\mathfrak{d}(\mathcal{O}))}{2^n [\widehat{O}_F^\times : \mathrm{Nr}(\widehat{\mathcal{O}}^\times)]} \prod_{\mathfrak{p} | \mathfrak{d}(\mathcal{O})} \frac{1 - \mathrm{N}(\mathfrak{p})^{-2}}{1 - e_{\mathfrak{p}}(\mathcal{O}) \, \mathrm{N}(\mathfrak{p})^{-1}}. \tag{3.12}$$

In particular, if $\mathcal{O}$ is residually unramified, then $\mathrm{Nr}(\widehat{\mathcal{O}}^\times) = \widehat{O}_F^\times$, and

$$\mathrm{Mass}^1(\mathcal{O}) = \frac{|\zeta_F(-1)| \, \mathrm{N}(\mathfrak{d}(\mathcal{O}))}{2^n} \prod_{\mathfrak{p} | \mathfrak{d}(\mathcal{O})} \frac{1 - \mathrm{N}(\mathfrak{p})^{-2}}{1 - e_{\mathfrak{p}}(\mathcal{O}) \, \mathrm{N}(\mathfrak{p})^{-1}}. \tag{3.13}$$

For the last part of this section, we assume that $\mathcal{O}$ is residually unramified and write down a formula for $h_{\mathrm{sc}}(\mathcal{O})$. Keep the notation of Theorems 1.1 and 1.3. In particular, $\mathscr{B}$ denotes the finite set of CM O_F-orders B with $w(B) > 1$, where $w(B) = [B^\times : O_F^\times]$. Let $M(B), \Delta(B, \mathcal{O}), s(B, \mathcal{O})$ be as defined in (1.4), (1.9), (1.10) respectively.

Theorem 3.2. *Suppose that $\mathcal{O}$ is residually unramified. Then*

$$h_{\mathrm{sc}}(\mathcal{O}) = \mathrm{Mass}_{\mathrm{sc}}(\mathcal{O}) + \frac{1}{2h^+(F)} \sum_{B \in \mathscr{B}} 2^{s(B,\mathcal{O})} \Delta(B, \mathcal{O})(w(B) - 1) M(B), \tag{3.14}$$

where $\mathrm{Mass}_{\mathrm{sc}}(\mathcal{O})$ *can be computed by* (3.8).

If further D is ramified at some finite prime of F, then $h_{\mathrm{sc}}(\mathcal{O})$ depends only on the genus of $\mathcal{O}$, and it is given by

$$h_{\mathrm{sc}}(\mathcal{O}) = \mathrm{Mass}_{\mathrm{sc}}(\mathcal{O}) + \frac{1}{2h^+(F)} \sum_{B \in \mathscr{B}} (w(B) - 1) M(B). \tag{3.15}$$

In this case every spinor class of locally principal right $\mathcal{O}$-ideals contains the same number of ideal classes, so $h(\mathcal{O}) = h^+(F) h_{\mathrm{sc}}(\mathcal{O})$.

Proof. Formula (3.15) is just a specialized version of (3.14); see a remark below Proposition 2.8. It is clear from (3.8) and (3.15) that $h_{\mathrm{sc}}(\mathcal{O})$ depends only on the genus of $\mathcal{O}$ when D is ramified at some finite place of F. Combining with (2.11) and (2.13), we see that the ideal classes are equi-distributed among the spinor classes, and hence $h(\mathcal{O}) = h^+(F) h_{\mathrm{sc}}(\mathcal{O})$.

The proof of (3.14) follows exactly the same argument as those for [40, Corollary V.2.5] and [44, Theorem 1.5], so we merely provide a sketch here. Let $h = |\mathrm{Cl}(\mathcal{O})|$, and $I_1, \cdots, I_h$ be a complete set of representatives for the full ideal class set $\mathrm{Cl}(\mathcal{O})$. Put $r = h_{\mathrm{sc}}(\mathcal{O})$, and arrange the ideals so that $[I_i] \in \mathrm{Cl}_{\mathrm{sc}}(\mathcal{O})$ for each $1 \leq i \leq r$. For every integral ideal $\mathfrak{a} \subseteq O_F$, there is an $h \times h$ integral matrix $\mathfrak{B}(\mathfrak{a}) = (\mathfrak{B}_{ij}(\mathfrak{a}))$ called the *Brandt matrix attached to* $\mathfrak{a}$ as in [44, Definition 3.1.3]. Each entry $\mathfrak{B}_{ij}(\mathfrak{a}) \in \mathbb{Z}$ is non-negative, and all the diagonal entries

$\mathfrak{B}_{ii}(\mathfrak{a})=0$ unless $\mathfrak{a}$ is principal and generated by a totally positive element. We define the *spinor Brandt matrix attached to* $\mathfrak{a}$ as

$$\mathfrak{B}_{\mathrm{sc}}(\mathfrak{a}):=(\mathfrak{B}_{ij}(\mathfrak{a}))_{1\le i,j\le r}. \tag{3.16}$$

In other words, $\mathfrak{B}_{\mathrm{sc}}(\mathfrak{a})$ is just the upper-left $(r\times r)$-block of $\mathfrak{B}(\mathfrak{a})$. Suppose that $\mathfrak{a}\subseteq O_F$ is a principal integral ideal generated by a totally positive element a. Fix a complete set $\mathscr{S}=\{\varepsilon_1,\cdots,\varepsilon_s\}$ of representatives for the finite elementary 2-group $O_{F,+}^\times/O_F^{\times 2}$. For each CM O_F-order B, we define a finite set

$$T_{B,\mathfrak{a}}:=\{x\in B\smallsetminus O_F\mid \mathrm{N}_{K/F}(x)=\varepsilon a \text{ for some } \varepsilon\in\mathscr{S}\}, \tag{3.17}$$

where K is the fractional field of B. Let $\mathscr{B}_\mathfrak{a}$ be the set of CM O_F-orders B with $T_{B,\mathfrak{a}}\neq\emptyset$. For example, if $\mathfrak{a}=O_F$, then $\mathscr{B}_\mathfrak{a}$ coincides with the set $\mathscr{B}$ considered above. In general, $\mathscr{B}_\mathfrak{a}$ is always a finite set, which can be proved in a similar way as the finiteness of $\mathscr{B}^1$ as explained right after Remark 4.3 in the next section. Mimicking the proof of the Eichler trace formula [40, Proposition 2.4] (cf. [44, Theorem 3.3.7]), we obtain a trace formula for $\mathrm{Tr}(\mathfrak{B}_{\mathrm{sc}}(\mathfrak{a}))$ as follows:

$$\mathrm{Tr}(\mathfrak{B}_{\mathrm{sc}}(\mathfrak{a}))=\delta_\mathfrak{a}\,\mathrm{Mass}_{\mathrm{sc}}(\mathcal{O})+\frac{1}{4h^+(F)}\sum_{B\in\mathscr{B}_\mathfrak{a}}2^{s(B,\mathcal{O})}\Delta(B,\mathcal{O})M(B)|T_{B,\mathfrak{a}}|, \tag{3.18}$$

where $\delta_\mathfrak{a}$ takes the value 1 or 0 depending on whether $\mathfrak{a}$ is the square of a principal ideal or not. Indeed, (3.18) can be proved in exactly the same way as [44, Theorem 3.3.7]. The only adjustment needed is to apply the spinor mass formula (3.3) and the spinor trace formula (2.14) when summing up the diagonal entries $\mathfrak{B}_{ii}(\mathfrak{a})$ in [44, (3.15)]. This is precisely the place where we have used the residually unramified assumption on $\mathcal{O}$. Lastly, note that $\mathfrak{B}_{ii}(\mathfrak{a})=1$ for every $1\le i\le h$ when $\mathfrak{a}=O_F$, so $h_{\mathrm{sc}}(\mathcal{O})=\mathrm{Tr}(\mathfrak{B}_{\mathrm{sc}}(O_F))$. It has been shown in the proof of [44, Corollary 3.3.8] that $|T_{B,O_F}|=2(w(B)-1)$. The class number formula (3.14) is now a direct consequence of (3.18). □

4. The class number formula for $h^1(\mathcal{O})$

Keep the notation and the assumptions of the previous section. In particular, F is a totally real field, and D is a totally definite quaternion F-algebra. Recall from (1.8) that $\mathscr{B}^1$ denotes the finite set of CM O_F-orders B with $|\boldsymbol{\mu}(B)|>2$, where $\boldsymbol{\mu}(B)$ is the group of roots of unity in B. The main result of this section is the following theorem, which connects the class number $h^1(\mathcal{O})$ with the left hand side of the spinor trace formula (2.14).

Theorem 4.1. *For any arbitrary O_F-order $\mathcal{O}$ in D, we have*

$$\begin{aligned}h^1(\mathcal{O})=&2\,\mathrm{Mass}^1(\mathcal{O})\\ &+\frac{u(\mathcal{O})}{4}\sum_{B\in\mathscr{B}^1}\frac{(|\boldsymbol{\mu}(B)|-2)}{w(B)}\sum_{[I]\in\mathrm{Cl}_{\mathrm{sc}}(\mathcal{O})}m(B,\mathcal{O}_l(I),\mathcal{O}_l(I)^\times),\end{aligned} \tag{4.1}$$

where $\mathrm{Mass}^1(\mathcal{O})$ *can be computed by* (3.12), *and* $u(\mathcal{O})=[(O_{F,+}^\times\cap\mathrm{Nr}(\widehat{\mathcal{O}}^\times)):O_F^{\times 2}]$ *as in* (3.6).

Recall that if $\mathcal{O}$ is residually unramified, then $\mathrm{Nr}(\widehat{\mathcal{O}}^\times)=\widehat{O}_F^\times$ by (2.3), and hence $u(\mathcal{O})=[O_{F,+}^\times:O_F^{\times 2}]$. In particular $h^+(F)=h(F)u(\mathcal{O})$ by (2.16). Combining Theorem 4.1 with Proposition 2.8, we immediately obtain the following corollary.

Corollary 4.2. *Suppose that $\mathcal{O}$ is residually unramified. Then*

$$h^1(\mathcal{O}) = 2\,\mathrm{Mass}^1(\mathcal{O}) + \frac{1}{4h(F)} \sum_{B\in\mathscr{B}^1} 2^{s(B,\mathcal{O})}\Delta(B,\mathcal{O})(|\boldsymbol{\mu}(B)|-2)M(B), \tag{4.2}$$

where $\mathrm{Mass}^1(\mathcal{O})$ *can be computed by* (3.13)*, and* $s(B,\mathcal{O}), \Delta(B,\mathcal{O}), M(B)$ *are given in* (1.10) (1.9)*,* (1.4) *respectively.*

If further D is ramified at some finite prime of F, then $h^1(\mathcal{O})$ depends only on the genus of $\mathcal{O}$, and it is given by

$$h^1(\mathcal{O}) = 2\,\mathrm{Mass}^1(\mathcal{O}) + \frac{1}{4h(F)} \sum_{B\in\mathscr{B}^1} (|\boldsymbol{\mu}(B)|-2)M(B). \tag{4.3}$$

Remark 4.3. If we drop the assumption that $\mathcal{O}$ is residually unramified, and assume instead that at least one of the following conditions holds for every CM O_F-order $B \in \mathscr{B}^1$:

(i) $s(B,\mathcal{O}) = 0$;

(ii) $m_{\mathfrak{p}}(B) = 0$ for some finite prime $\mathfrak{p}$ of F;

(iii) B is optimally spinor selective for the genus of $\mathcal{O}$;

then we can still compute $h^1(\mathcal{O})$ by combining Theorem 4.1 with the slightly more general version of spinor trace formula in [47, Proposition 4.3]. As mentioned in Remark 2.9, condition (iii) is difficult to check in general, but there are known cases beyond the residually unramified case in [30, Theorem 2.6].

The rest of this section is devoted to the proof of Theorem 4.1, so henceforth $\mathcal{O}$ is an arbitrary O_F-order in D. We first give a short analysis of the set $\mathscr{B}^1$, which will prove its finiteness as a by-product. Let B be a member of $\mathscr{B}^1$ with fraction field K. For any $\gamma \in \boldsymbol{\mu}(B) \smallsetminus \{\pm 1\}$, the minimal polynomial of γ over F is of the form

$$P_b(T) := T^2 - bT + 1 \in F[T], \tag{4.4}$$

where $b = \mathrm{Tr}_{K/F}(\gamma) \in O_F$. Necessarily $4 - b^2 \in F_+^\times$ since K/F is a CM-extension. This leads to the consideration of the set

$$\mathcal{T} := \{b \in O_F \mid 4 - b^2 \in F_+^\times\}, \tag{4.5}$$

which is finite since O_F is discrete in $F \otimes_{\mathbb{Q}} \mathbb{R}$. For each $b \in \mathcal{T}$, let $\mathcal{K}_b := F[T]/(P_b(T))$ and $\bar{T}$ be the canonical image of T in $\mathcal{K}_b$. As b ranges over $\mathcal{T}$, the ordered pair $(\mathcal{K}_b, \bar{T})$ ranges over all CM-extensions K/F with $|\boldsymbol{\mu}(K)| > 2$ together with a marked root of unity of order > 2. Let $\mathscr{B}_b^1$ be the finite set of O_F-orders in $\mathcal{K}_b$ as follows:

$$\mathscr{B}_b^1 := \{B \subset \mathcal{K}_b \mid O_F[\bar{T}] \subseteq B \subseteq O_{\mathcal{K}_b}\}. \tag{4.6}$$

Clearly, $\mathscr{B}^1 := \bigcup_{b\in\mathcal{T}} \mathscr{B}_b^1$, which implies that $\mathscr{B}^1$ is a finite set. In fact, the fiber of the canonical map

$$\coprod_{b\in\mathcal{T}} \mathscr{B}_b^1 \to \mathscr{B}^1 \tag{4.7}$$

over each $B \in \mathscr{B}^1$ has exactly $(|\boldsymbol{\mu}(B)| - 2)/2$ elements.

To prove Theorem 4.1, we apply the *Selberg trace formula* for compact quotient. See [5, §1] and [31, §5] for brief introductions. Let H be a locally compact totally disconnected topological group (a group of *td-type* as in [11, §1]). We further assume that H is unimodular with a Haar measure dx. If H_1 is a unimodular closed subgroup of H with Haar measure

dy, then by an abuse of notation, we still write dx for the induced right H-invariant measure [28, Corollary 4, §III.4] on the homogeneous space $H_1\backslash H$, which is characterized by the following integration formula

$$\int_H f(x)dx = \int_{H_1\backslash H}\int_{H_1} f(yx)dydx, \qquad \forall f \in C_c^\infty(H). \tag{4.8}$$

Here $C_c^\infty(H)$ denotes the space of locally constant $\mathbb{C}$-valued functions on H with compact support. If $V \subseteq H$ is an open subset of H, we write[2] $\mathrm{Vol}(V, H_1\backslash H) \in [0,\infty]$ for the volume of the canonical image of V in $H_1\backslash H$. In other words,

$$\mathrm{Vol}(V, H_1\backslash H) := \int_{H_1\backslash H} \mathbb{1}_{H_1V}(x)dx. \tag{4.9}$$

Here $\mathbb{1}_{H_1V}(x)$ denotes the characteristic function on H of the open subset $H_1V \subseteq H$, which descends to the characteristic function on $H_1\backslash H$ for the image of V. Now let $H_2 \subseteq H_1$ be another unimodular closed subgroup with Haar measure dz. Suppose that $V \subseteq H$ is an open compact subgroup. Then for any $x \in H$, we have

$$\mathrm{Vol}(xV, H_2\backslash H) = \mathrm{Vol}(H_1 \cap xVx^{-1}, H_2\backslash H_1)\cdot \mathrm{Vol}(xV, H_1\backslash H). \tag{4.10}$$

For example, if H_2 is the trivial group, then we get

$$\mathrm{Vol}(xV, H_1\backslash H) = \frac{\mathrm{Vol}(V, H)}{\mathrm{Vol}(H_1 \cap xVx^{-1}, H_1)}. \tag{4.11}$$

If $x \in H_1$, then (4.10) simplifies into

$$\mathrm{Vol}(xV, H_2\backslash H) = \mathrm{Vol}(x(H_1 \cap V), H_2\backslash H_1)\cdot \mathrm{Vol}(V, H_1\backslash H). \tag{4.12}$$

Suppose that Γ is a discrete cocompact subgroup of H. Denote by $L^2(\Gamma\backslash H)$ the Hilbert space of $\mathbb{C}$-valued square-integrable functions on $\Gamma\backslash H$. For any $f \in C_c^\infty(H)$, we form an operator $R(f) : L^2(\Gamma\backslash H) \to L^2(\Gamma\backslash H)$ by

$$(R(f)\phi)(y) = \int_H f(x)\phi(yx)dx = \int_H f(y^{-1}x)\phi(x)dx \tag{4.13}$$

for all $y \in H$ and $\phi \in L^2(\Gamma\backslash H)$. Normalize the Haar measure on H so that $\mathrm{Vol}(U) = 1$ for a fixed open compact subgroup $U \subseteq H$. It is clear from (4.13) that $R(\mathbb{1}_U)$ is the projection onto the U-invariant subspace $L^2(\Gamma\backslash H)^U = L^2(\Gamma\backslash H/U)$. Therefore,

$$\mathrm{Tr}(\mathbb{1}_U) = \dim_{\mathbb{C}} L^2(\Gamma\backslash H/U) = |\Gamma\backslash H/U|. \tag{4.14}$$

For any $\gamma \in \Gamma$ and any subset S in H, we write S_γ for the centralizer of γ in S. Let $\{\gamma\}$ be the Γ-conjugacy class of $\gamma \in \Gamma$, and $\{\Gamma\}$ be the set of all conjugacy classes of Γ. Applying the Selberg trace formula [5, p. 9] to $f = \mathbb{1}_U$, we obtain

$$\mathrm{Tr}(\mathbb{1}_U) = \sum_{\{\gamma\}\in\{\Gamma\}} \int_{\Gamma_\gamma\backslash H} \mathbb{1}_U(x^{-1}\gamma x)dx. \tag{4.15}$$

Here each Γ_γ is equipped with the counting measure, and dx is the induced right H-invariant measure on $\Gamma_\gamma\backslash H$ as in (4.8). It is well known that the right hand side of (4.15) is a finite sum; see [49, Proposition 8.1].

[2] Admittedly, this notation is rather nonstandard. However, since we will be dealing with multiple homogeneous spaces soon, it might be helpful to keep track of the space at hand.

Applying (4.14) and (4.15) in the case that

$$H = \widehat{D}^1, \qquad U = \widehat{\mathcal{O}}^1, \qquad \Gamma = D^1, \tag{4.16}$$

we express $h^1(\mathcal{O})$ into a sum of integrals. Note that $\gamma \in D^1$ is central if and only if $\gamma = \pm 1$, in which case the summand in (4.15) corresponding to $\{\gamma\}$ reduces to $\mathrm{Vol}(D^1 \backslash \widehat{D}^1)$. From (3.11), the contribution of the central elements $\{\pm 1\} \subset D^1$ to $h^1(\mathcal{O})$ is precisely $2\,\mathrm{Mass}^1(\mathcal{O})$, the first term in the right hand side of (4.1). Thus the proof of Theorem 4.1 is reduced to matching the second line of (4.1) with the contributions of the noncentral classes in $\{D^1\}$.

Remark 4.4. Traditionally, the orbital integral in the Selberg trace formula (4.15) is expanded out a bit further and reads

$$\int_{\Gamma_\gamma \backslash H} \mathbb{1}_U(x^{-1}\gamma x)dx = \mathrm{Vol}(\Gamma_\gamma \backslash H_\gamma) \int_{H_\gamma \backslash H} \mathbb{1}_U(x^{-1}\gamma x)dx. \tag{4.17}$$

Here the volume $\mathrm{Vol}(\Gamma_\gamma \backslash H_\gamma)$ depends on the choice of the Haar measure on H_γ. However, as a whole the right hand side of (4.17) is independent of such choices. Suppose that $\gamma \neq \pm 1$, and let K be the CM-field $F(\gamma)$ generated by γ over F. Then K coincides with the centralizer of γ in D, so $\Gamma_\gamma = K^1$ and $H_\gamma = \widehat{K}^1$. Let O_K be the ring of integers of K. We normalize the Haar measure on $\widehat{K}^1$ so that its unique maximal open compact subgroup $\widehat{O}_K^1$ has volume one. Now it is a classical result of Takashi Ono [29] (simplified by Sasaki in [36, Theorem 3]) that

$$\mathrm{Vol}(K^1 \backslash \widehat{K}^1) = \frac{h(K/F)}{2^{t-1} Q_{K/F} |\boldsymbol{\mu}(K)|} = \frac{h(K/F)}{2^t [O_K^\times : O_F^\times]}. \tag{4.18}$$

Here the invariants in the above formula are as follows

- $h(K/F) = h(K)/h(F)$ is the relative class number of the CM-extension K/F [42, Theorem 4.10];
- t is the number of finite primes of F ramified in K;
- $Q_{K/F} := [O_K^\times : O_F^\times \boldsymbol{\mu}(K)]$ is the Hasse unit index [14, §13, p. 69], which takes value either 1 or 2 (see also [42, Theorem 4.12]).

An unsuspecting reader might expect that our method of proof for the formula of $h^1(\mathcal{O})$ is simply working out the finitely many orbital integrals $\int_{H_\gamma \backslash H} \mathbb{1}_U(x^{-1}\gamma x)dx$ term by term. There are a couple of reasons going against this approach, which require some explanations.

The reader may have already observed that the invariant t makes no appearance in our formula for $h^1(\mathcal{O})$. The cancellation does not happen at the spot when we multiply $\mathrm{Vol}(K^1 \backslash \widehat{K}^1)$ with $\int_{\widehat{K}^1 \backslash \widehat{D}^1} \mathbb{1}_U(x^{-1}\gamma x)$. For example, let $p \in \mathbb{N}$ be prime with $p > 3$. Let $F = \mathbb{Q}(\sqrt{p}\,)$ and $D := D_{\infty_1, \infty_2}$ be the unique totally definite quaternion F-algebra that is unramified at all the finite primes of F. Suppose that $\mathcal{O}$ is a maximal order in D that contains a root of unity $\gamma \in \mathcal{O}^1$ of order 3. Then $K = F(\sqrt{-3}\,)$, and $O_F[\gamma]$ coincides with O_K by [23, §7]. Thus $\mathcal{O} \cap K = O_F[\gamma] = O_K$, and $\mathscr{B}_b^1$ consists of the unique CM O_F-order O_K for $b = \mathrm{Tr}(\gamma) = -1$. Clearly,

$$t = \begin{cases} 1 & \text{if } p \equiv 2 \pmod 3; \\ 2 & \text{if } p \equiv 1 \pmod 3. \end{cases}$$

On the other hand, it can be shown that $\int_{\widehat{K}^1 \backslash \widehat{D}^1} \mathbb{1}_U(x^{-1}\gamma x)dx = 1$ in both cases for p. Already, we see that the invariant $t = 2$ does show up in the result of (4.17) when $p \equiv 1$

(mod 3). Curiously, γ and its complex conjugate $\bar{\gamma} \in K$ belong to the same D^1-conjugacy class if and only if $p \equiv 2 \pmod 3$ by Example 4.8. Thus if $p \equiv 1 \pmod 3$, we get more D^1-conjugacy classes of order 3, with each contributing less to $h^1(\mathcal{O})$, so the total contributions are balanced out and t disappears in the final formula for $h^1(\mathcal{O})$. This interesting phenomenon calls for an approach that dispels the mystery behind such cancellations.

Yet another reason for not taking the term-by-term approach is the presence of the optimal spinor selectivity symbols $\Delta(B, \mathcal{O})$ and $s(B, \mathcal{O})$ in the expected formula for $h^1(\mathcal{O})$. As we have discovered with concrete examples, as soon as one tries to compute the orbital integral $\int_{H_\gamma \backslash H} \mathbb{1}_U(x^{-1}\gamma x)dx$ by brute force, the selectivity theory inevitably shows up and get mixed with other volume calculations, making it more unmanageable. The new approach to be explained below will kill two birds with one stone: the invariant t never appears, and the selectivity theory is hidden within the summation $\sum_{[I]\in \mathrm{Cl}_{\mathrm{sc}}(\mathcal{O})} m(B, \mathcal{O}_l(I), \mathcal{O}_l(I)^\times)$ and only manifests itself when we apply the spinor trace formula at the very end.

We return to the proof of Theorem 4.1. Let $H = \widehat{D}^1, U = \widehat{\mathcal{O}}^1$ and $\Gamma = D^1$ be as in (4.16). We have already obtained the following formula:

$$h^1(\mathcal{O}) = \sum_{\{\gamma\}\in\{\Gamma\}} \int_{\Gamma_\gamma \backslash H} \mathbb{1}_U(x^{-1}\gamma x)dx, \tag{4.19}$$

and the contribution of the central elements $\{\pm 1\}$ has been shown to be the mass part of the desired formula (4.1). Assume that $\gamma \in D^1 \smallsetminus \{\pm 1\}$ for the rest of this section. By an abuse of notation, we write K_γ for the CM-field $F(\gamma)$, which is the centralizer of γ in D. Let us put

$$\widehat{\mathcal{E}}^1(\gamma) := \{x \in \widehat{D}^1 \mid x^{-1}\gamma x \in \widehat{\mathcal{O}}^1\}, \tag{4.20}$$

so that

$$\int_{\Gamma_\gamma \backslash H} \mathbb{1}_U(x^{-1}\gamma x)dx = \mathrm{Vol}(\widehat{\mathcal{E}}^1(\gamma), K_\gamma^1 \backslash \widehat{D}^1). \tag{4.21}$$

Since $\widehat{\mathcal{E}}^1(\gamma)$ is an open subset of $\widehat{D}^1$, the volume is nonzero if and only if $\widehat{\mathcal{E}}^1(\gamma) \neq \emptyset$. The latter condition implies that $b = \mathrm{Tr}(\gamma) \in \mathcal{T}$ as in (4.5), or equivalently, $\gamma \in \boldsymbol{\mu}(K_\gamma)$ by [42, Lemma 1.6]. For each $b \in \mathcal{T}$, we put

$$\Gamma(b) := \{\gamma \in D^1 \mid \mathrm{Tr}(\gamma) = b\}, \tag{4.22}$$

which forms a single $D^\times$-conjugacy class in D^1. Given $\gamma \in \Gamma(b)$, the F-embedding

$$\varphi_\gamma : \mathcal{K}_b \hookrightarrow D \qquad \bar{T} \mapsto \gamma, \tag{4.23}$$

identifies $\mathcal{K}_b$ with K_γ. For each $B \in \mathscr{B}_b^1$, we define

$$\widehat{\mathcal{E}}(\gamma, B) := \{g = (g_\mathfrak{p}) \in \widehat{D}^\times \mid K_\gamma \cap g\widehat{\mathcal{O}}g^{-1} = \varphi_\gamma(B)\} \tag{4.24}$$

and put $\widehat{\mathcal{E}}^1(\gamma, B) := \widehat{\mathcal{E}}(\gamma, B) \cap \widehat{D}^1$. There is a left K_γ^1-equivariant decomposition

$$\widehat{\mathcal{E}}^1(\gamma) = \bigsqcup_{B\in\mathscr{B}_b^1} \widehat{\mathcal{E}}^1(\gamma, B), \tag{4.25}$$

which implies that

$$\int_{\Gamma_\gamma\backslash H} \mathbb{1}_U(x^{-1}\gamma x)dx = \sum_{B\in\mathscr{B}_b^1} \mathrm{Vol}(\widehat{\mathcal{E}}^1(\gamma,B), K_\gamma^1\backslash\widehat{D}^1). \tag{4.26}$$

Grouping together all those Γ-conjugacy classes $\{\gamma\}$ with the same minimal polynomial in the right hand side of (4.19), we obtain

$$\begin{aligned} h^1(\mathcal{O}) &= 2\,\mathrm{Mass}^1(\mathcal{O}) + \sum_{b\in\mathcal{T}} \sum_{\{\gamma\}\in\{\Gamma(b)\}} \int_{\Gamma_\gamma\backslash H} \mathbb{1}_U(x^{-1}\gamma x)dx \\ &= 2\,\mathrm{Mass}^1(\mathcal{O}) + \sum_{b\in\mathcal{T}} \sum_{B\in\mathscr{B}_b^1} \sum_{\{\gamma\}\in\{\Gamma(b)\}} \mathrm{Vol}(\widehat{\mathcal{E}}^1(\gamma,B), K_\gamma^1\backslash\widehat{D}^1). \end{aligned} \tag{4.27}$$

Proposition 4.5. *For each $b\in\mathcal{T}$ and $B\in\mathscr{B}_b^1$, we have*

$$\sum_{\{\gamma\}\in\{\Gamma(b)\}} \mathrm{Vol}(\widehat{\mathcal{E}}^1(\gamma,B), K_\gamma^1\backslash\widehat{D}^1) = \frac{u(\mathcal{O})}{2w(B)} \sum_{[I]\in\mathrm{Cl}_{\mathrm{sc}}(\mathcal{O})} m(B,\mathcal{O}_l(I),\mathcal{O}_l(I)^\times). \tag{4.28}$$

Note that the right hand side of (4.28) depends only on B and not on $b\in\mathcal{T}$. For each $B\in\mathscr{B}^1$, the number of $b\in\mathcal{T}$ with $B\in\mathscr{B}_b^1$ is precisely $(|\boldsymbol{\mu}(B)|-2)/2$. Thus Theorem 4.1 follows directly from Proposition 4.5.

Proposition 4.5 is nonintuitive in a couple of ways. The summation on the right hand side of (4.28) is over $\mathrm{Cl}_{\mathrm{sc}}(\mathcal{O})$. As a double coset space, it is given by $D^\times\backslash\big(D^\times\widehat{D}^1\widehat{\mathcal{O}}^\times\big)/\widehat{\mathcal{O}}^\times$ rather than $D^1\backslash\widehat{D}^1/\widehat{\mathcal{O}}^1$. This discrepancy is further reflected in the adelic description of the summation itself, which we recall below. For each $\gamma\in\Gamma(b)$ and each $B\in\mathscr{B}_b^1$, consider the set $\widehat{\mathcal{E}}(\gamma,B)\subset\widehat{D}^\times$ defined in (4.24). Note that $\widehat{\mathcal{E}}(\gamma,B)$ is left translation invariant by $\widehat{K}_\gamma^\times$ and right translation invariant by the normalizer $\mathcal{N}(\widehat{\mathcal{O}})\subset\widehat{D}^\times$. For any other $\gamma'\in\Gamma(b)$, there exists $\alpha\in D^\times$ such that $\gamma'=\alpha^{-1}\gamma\alpha$. A straightforward computation shows that

$$\widehat{\mathcal{E}}(\gamma',B) = \widehat{\mathcal{E}}(\alpha^{-1}\gamma\alpha,B) = \alpha^{-1}\widehat{\mathcal{E}}(\gamma,B). \tag{4.29}$$

Let us define

$$\widehat{\mathcal{E}}_{\mathrm{sc}}(\gamma,B) := \widehat{\mathcal{E}}(\gamma,B)\cap D^\times\widehat{D}^1\widehat{\mathcal{O}}^\times = \big(\widehat{\mathcal{E}}(\gamma,B)\cap D^\times\widehat{D}^1\big)\widehat{\mathcal{O}}^\times. \tag{4.30}$$

Clearly, $\widehat{\mathcal{E}}_{\mathrm{sc}}(\gamma,B)$ is left translation invariant by $K_\gamma^\times$ and right translation invariant by $\widehat{\mathcal{O}}^\times$. From [47, Lemma 4.5], for any $\gamma\in\Gamma(b)$ we have

$$\sum_{[I]\in\mathrm{Cl}_{\mathrm{sc}}(\mathcal{O})} m(B,\mathcal{O}_l(I),\mathcal{O}_l(I)^\times) = |K_\gamma^\times\backslash\widehat{\mathcal{E}}_{\mathrm{sc}}(\gamma,B)/\widehat{\mathcal{O}}^\times|. \tag{4.31}$$

Thus to prove (4.28), it is enough to fix one $\gamma_0\in\Gamma(b)$ and show that

$$\frac{u(\mathcal{O})}{2w(B)}|K_{\gamma_0}^\times\backslash\widehat{\mathcal{E}}_{\mathrm{sc}}(\gamma_0,B)/\widehat{\mathcal{O}}^\times| = \sum_{\{\gamma\}\in\{\Gamma(b)\}} \mathrm{Vol}(\widehat{\mathcal{E}}^1(\gamma,B), K_\gamma^1\backslash\widehat{D}^1). \tag{4.32}$$

Comparing both sides of (4.32), we see that the double coset space on the left hand side involves subsets of $\widehat{D}^\times$, while the right hand side only involves those of $\widehat{D}^1$. This difference in nature of the two sides of (4.32) is what causes the technicality in its proof.

To bridge such differences, we forget the notation in (4.16) and study another set of groups previously considered in (3.5):

$$H := \widehat{D}^\times/\widehat{O}_F^\times, \qquad U := \widehat{\mathcal{O}}^\times/\widehat{O}_F^\times, \qquad \Omega := D^\times\widehat{O}_F^\times/\widehat{O}_F^\times \simeq D^\times/O_F^\times. \tag{4.33}$$

As explained in the proof of Lemma 3.1, H is a unimodular group of td-type, and Ω is discrete cocompact in H. We equip Ω with the counting measure, and normalize the Haar measure on H so that $\mathrm{Vol}(U) = 1$. The image of $\widehat{D}^1$ inside H is $(\widehat{D}^1\widehat{O}_F^\times)/\widehat{O}_F^\times$, which is canonically isomorphic to $\widehat{D}^1/(\widehat{D}^1 \cap \widehat{O}_F^\times)$. Let us put

$$C := \widehat{D}^1 \cap \widehat{O}_F^\times = \prod_{\mathfrak{p}} \{\pm 1\}, \tag{4.34}$$

where the product ranges over all finite primes of F. Clearly, C is a compact subgroup of $\widehat{\mathcal{O}}^1$, and $D^1 \cap C = \{\pm 1\}$. Consider the following subgroups of H:

$$D^\dagger := D^1/\{\pm 1\}, \qquad \widehat{D}^\dagger := \widehat{D}^1/C, \qquad \widehat{\mathcal{O}}^\dagger := \widehat{\mathcal{O}}^1/C.$$

There is a canonical bijection between double coset spaces as follows

$$D^1\backslash\widehat{D}^1/\widehat{\mathcal{O}}^1 \xrightarrow{\simeq} D^\dagger\backslash\widehat{D}^\dagger/\widehat{\mathcal{O}}^\dagger. \tag{4.35}$$

We shall make no explicit usage of this bijection below, but this is one of the motivations for our proof of Proposition 4.5.

Lemma 4.6. $(D^\times\widehat{D}^1) \cap \widehat{O}_F^\times = O_F^\times C$.

Proof. Clearly, $(D^\times\widehat{D}^1) \cap \widehat{O}_F^\times \supseteq O_F^\times C$. For any $x \in (D^\times\widehat{D}^1) \cap \widehat{O}_F^\times$, we have

$$\mathrm{Nr}(x) \in \mathrm{Nr}(D^\times) \cap \widehat{O}_F^{\times 2} = F_+^\times \cap \widehat{O}_F^{\times 2} = O_F^{\times 2},$$

where the last equality follows from Hasse-Minkowski Theorem [33, Theorem 1.6]. Pick $\xi \in O_F^\times$ such that $\xi^2 = \mathrm{Nr}(x)$. Then $\xi^{-1}x \in \widehat{D}^1 \cap \widehat{O}_F^\times = C$. It follows that $(D^\times\widehat{D}^1) \cap \widehat{O}_F^\times \subseteq O_F^\times C$, and the lemma is proved. □

An important link between the groups $\widehat{D}^\dagger \subseteq H$ will be the following intermediate group H_1 defined below.

Lemma 4.7. *The group*

$$H_1 := \big(D^\times\widehat{D}^1\widehat{O}_F^\times\big)/\widehat{O}_F^\times \simeq (D^\times\widehat{D}^1)/(O_F^\times C) \tag{4.36}$$

is a unimodular closed normal subgroup of H, and it contains

$$\widehat{\mathcal{O}}^\dagger = (\widehat{\mathcal{O}}^1\widehat{O}_F^\times)/\widehat{O}_F^\times \simeq \widehat{\mathcal{O}}^1/C \tag{4.37}$$

as an open compact subgroup. In particular, $\widehat{D}^\dagger$ is open in H_1 as well.

Proof. Since the quotient $\widehat{D}^\times/\widehat{D}^1 \simeq \widehat{F}^\times$ is abelian, any subgroup of H containing $\widehat{D}^\dagger$ is normal, and the reduced norm map induces an isomorphism

$$H_1\backslash H \simeq \widehat{F}^\times/(F_+^\times\widehat{O}_F^{\times 2}). \tag{4.38}$$

By [33, Theorem 5.2], there is a compact fundamental set Z for D^1 in $\widehat{D}^1$. Then $H_1 = \Omega\widetilde{Z}$, where $\widetilde{Z}$ denotes the canonical image of Z in H, and hence H_1 is closed by [6, Lemma 1, §X.2]. Now it follows from [7, Proposition B.2.2] that H_1 is unimodular.

Clearly, $\widehat{\mathcal{O}}^\dagger$ is a compact subgroup of H_1. To show that it is open in H_1, it is enough to show that $\widehat{\mathcal{O}}^\dagger$ has finite index in the open subgroup $H_1 \cap U$, where $U = \widehat{\mathcal{O}}^\times / \widehat{O}_F^\times$ as in (4.33). We calculate

$$\begin{aligned}
&[H_1 \cap U : \widehat{\mathcal{O}}^\dagger] = [\big((D^\times \widehat{D}^1 \widehat{O}_F^\times) \cap \widehat{\mathcal{O}}^\times\big) : \widehat{\mathcal{O}}^1 \widehat{O}_F^\times] \\
=&[\big((D^\times \widehat{D}^1) \cap \widehat{\mathcal{O}}^\times\big)\widehat{O}_F^\times : \widehat{\mathcal{O}}^1 \widehat{O}_F^\times] = [\big((D^\times \widehat{D}^1) \cap \widehat{\mathcal{O}}^\times\big) : \big((D^\times \widehat{D}^1) \cap \widehat{\mathcal{O}}^\times \cap \widehat{\mathcal{O}}^1 \widehat{O}_F^\times\big)] \\
=&[\big((D^\times \widehat{D}^1) \cap \widehat{\mathcal{O}}^\times\big) : \widehat{\mathcal{O}}^1 O_F^\times] = [\mathrm{Nr}\left((D^\times \widehat{D}^1) \cap \widehat{\mathcal{O}}^\times\right) : \mathrm{Nr}(\widehat{\mathcal{O}}^1 O_F^\times)] \\
=&[\big(O_{F,+}^\times \cap \mathrm{Nr}(\widehat{\mathcal{O}}^\times)\big) : O_F^{\times 2}] = u(\mathcal{O}) < \infty. \qquad \square
\end{aligned}$$

We normalize the Haar measure on H_1 so that $\mathrm{Vol}(\widehat{\mathcal{O}}^\dagger) = 1$. From (4.11),

$$\mathrm{Vol}(\mathrm{Nr}(\widehat{\mathcal{O}}^\times), \widehat{F}^\times/(F_+^\times \widehat{O}_F^{\times 2})) = \mathrm{Vol}(U, H_1\backslash H) = 1/u(\mathcal{O}) \tag{4.39}$$

with respect to the induced measure on $H_1\backslash H$.

Let us fix $\gamma_0 \in \Gamma(b)$ and put $K_0 := K_{\gamma_0} = F(\gamma_0)$. There is a bijection

$$K_0^\times \backslash D^\times \to \Gamma(b), \qquad K_0^\times \alpha \mapsto \alpha^{-1}\gamma_0\alpha, \tag{4.40}$$

which gives rise to a bijection

$$K_0^\times \backslash D^\times / D^1 \to \{\Gamma(b)\}. \tag{4.41}$$

On the other hand, the reduced norm map induces a bijection

$$K_0^\times \backslash D^\times / D^1 \xrightarrow[\mathrm{Nr}]{\simeq} F_+^\times / \mathrm{Nr}(K_0^\times). \tag{4.42}$$

Piecing together the above bijections, we obtain

$$F_+^\times / \mathrm{Nr}(K_0^\times) \simeq \{\Gamma(b)\}. \tag{4.43}$$

According to Lemma 4.6, we have

$$D^\times \cap (\widehat{D}^1 \widehat{O}_F^\times) = D^\times \cap \big(\widehat{D}^1(O_F^\times C)\big) = D^\times \cap (\widehat{D}^1 O_F^\times) = (D^\times \cap \widehat{D}^1) O_F^\times = D^1 O_F^\times.$$

It follows that there is a canonical isomorphism

$$K_0^\times \backslash \big(D^\times \widehat{D}^1 \widehat{O}_F^\times\big)/(\widehat{D}^1 \widehat{O}_F^\times) \simeq K_0^\times \backslash D^\times/(D^1 O_F^\times) = K_0^\times \backslash D^\times / D^1. \tag{4.44}$$

Example 4.8. Let F be a real quadratic field, and $D = D_{\infty_1,\infty_2}$ be the unique totally definite quaternion F-algebra that is unramified at all the finite primes of F. Let $\gamma \in D^1$ be an element of order 3, and $\bar\gamma := \mathrm{Tr}(\gamma) - \gamma$ be its image under the canonical involution. We claim that γ and $\bar\gamma$ are D^1-conjugate if and only if 3 is non-split in F. Since the set of elements of order 3 forms a single $D^\times$-conjugacy class, it is enough to check this for one γ_0 of order 3. Given $a, b \in F^\times$, we write $\left(\frac{a,b}{F}\right)$ for the quaternion F-algebra with F-basis $\{1, i, j, ij\}$ subject to the following multiplication rules

$$i^2 = a, \quad j^2 = b, \quad \text{and} \quad ij = -ji.$$

First, suppose that 3 is non-split in F. Then $D = \left(\frac{-1,-3}{F}\right)$, and $\gamma_0 := (-1+j)/2 \in D^1$ is an element of order 3. Clearly, γ_0 and $\bar\gamma_0$ are D^1-conjugate since $\bar\gamma_0 = i\gamma_0 i^{-1}$. Next, suppose that 3 is split in F. Let $\ell \in \mathbb{N}$ be a prime with $\ell \equiv 2 \pmod 3$. Then $\left(\frac{-\ell,-3}{\mathbb{Q}}\right)$ is the unique quaternion $\mathbb{Q}$-algebra ramified precisely at ℓ and ∞. Pick ℓ so that it is inert in F. Then $D = \left(\frac{-\ell,-3}{F}\right)$. Take $\gamma_0 = (-1+j)/2 \in D^1$ as before. Once again, we have $\bar\gamma_0 = i\gamma_0 i^{-1}$, except that this time $\mathrm{Nr}(i) = \ell$. From (4.43), if $\bar\gamma_0$ and γ_0 are D^1-conjugate, then $\ell \in \mathrm{Nr}(K_0^\times)$, where $K_0 = F(\gamma_0) \simeq F(\sqrt{-3}\,)$. On the other hand, let $\mathfrak{p}$ be one of the primes of O_F lying above 3, then $F_\mathfrak{p} = \mathbb{Q}_3$, and $(K_0)_\mathfrak{p} = \mathbb{Q}_3(\sqrt{-3}\,)$. Already ℓ is not a local norm at $\mathfrak{p}$ since $\ell \equiv 2 \pmod 3$. Therefore, $\bar\gamma_0$ and γ_0 are not D^1-conjugate when 3 is split in F. Our claim is verified.

We return to the general case where both F and D are arbitrary.

Proof of Proposition 4.5. As explained in (4.32), it is enough to show that

$$\frac{u(\mathcal{O})}{2w(B)}|K_0^\times\backslash\widehat{\mathcal{E}}_{\mathrm{sc}}(\gamma_0, B)/\widehat{\mathcal{O}}^\times| = \sum_{\{\gamma\}\in\{\Gamma(b)\}} \mathrm{Vol}(\widehat{\mathcal{E}}^{\,1}(\gamma, B), K_\gamma^1\backslash\widehat{D}^1)$$

for a fixed $\gamma_0 \in \Gamma(b)$. To bring the groups $H = \widehat{D}^\times/\widehat{O}_F^\times$, $U = \widehat{\mathcal{O}}^\times/\widehat{O}_F^\times$ and $\Omega = D^\times/O_F^\times$ into relevancy, first note that there is a canonical bijection

$$K_0^\times\backslash\widehat{\mathcal{E}}_{\mathrm{sc}}(\gamma_0, B)/\widehat{\mathcal{O}}^\times \simeq H_2\backslash\widehat{\mathcal{E}}^\sharp_{\mathrm{sc}}(\gamma_0, B)/U, \tag{4.45}$$
$$\text{where}\quad \widehat{\mathcal{E}}^\sharp_{\mathrm{sc}}(\gamma_0, B) := \widehat{\mathcal{E}}_{\mathrm{sc}}(\gamma_0, B)/\widehat{O}_F^\times \quad\text{and}\quad H_2 := (K_0^\times\widehat{O}_F^\times)/\widehat{O}_F^\times = K_0^\times/O_F^\times \subset \Omega.$$

Equip H_2 with the counting measure and let $H_1 = D^\times\widehat{D}^1/(O_F^\times C) \subseteq H$ be as in (4.36). From the definition of $\widehat{\mathcal{E}}_{\mathrm{sc}}(\gamma_0, B)$ in (4.30), there exists a complete set of representatives $y_1, \cdots, y_r \in H_1$ for the double coset space in (4.45) so that

$$H \supset \widehat{\mathcal{E}}^\sharp_{\mathrm{sc}}(\gamma_0, B) = \bigsqcup_{i=1}^r H_2 y_i U. \tag{4.46}$$

Thanks to (4.11), we have

$$\mathrm{Vol}(y_iU, H_2\backslash H) = \frac{\mathrm{Vol}(U, H)}{\mathrm{Vol}(H_2 \cap y_iUy_i^{-1}, H_2)} = \frac{1}{\mathrm{Vol}(B^\times/O_F^\times, H_2)} = \frac{1}{w(B)} \tag{4.47}$$

for each $1 \le i \le r$. It follows that

$$\frac{1}{w(B)}|H_2\backslash\widehat{\mathcal{E}}^\sharp_{\mathrm{sc}}(\gamma_0, B)/U| = \mathrm{Vol}(\widehat{\mathcal{E}}^\sharp_{\mathrm{sc}}(\gamma_0, B), H_2\backslash H). \tag{4.48}$$

On the other hand, we apply (4.12) and (4.39) to obtain

$$\mathrm{Vol}(y_iU, H_2\backslash H) = \frac{1}{u(\mathcal{O})}\mathrm{Vol}(y_i(H_1 \cap U), H_2\backslash H_1). \tag{4.49}$$

Summing both sides over $1 \le i \le r$, we get

$$\begin{aligned}
&u(\mathcal{O})\,\mathrm{Vol}\left(\widehat{\mathcal{E}}^\sharp_{\mathrm{sc}}(\gamma_0, B), H_2\backslash H\right)\\
&= \mathrm{Vol}\left(\big(\widehat{\mathcal{E}}_{\mathrm{sc}}(\gamma_0, B) \cap D^\times\widehat{D}^1\widehat{O}_F^\times\big)/\widehat{O}_F^\times,\ H_2\backslash H_1\right)\\
&= \mathrm{Vol}\left(\big(\widehat{\mathcal{E}}(\gamma_0, B) \cap D^\times\widehat{D}^1\widehat{O}_F^\times\big)/\widehat{O}_F^\times,\ H_2\backslash H_1\right).
\end{aligned} \tag{4.50}$$

Here in the last step we have plugged in the definition of $\widehat{\mathcal{E}}_{\mathrm{sc}}(\gamma_0, B)$ as in (4.30).

Let $\{\alpha_j\}_{j\in A} \subset D^\times$ be a complete set of representatives for $K_0^\times\backslash D^\times/D^1$, where $A := F_+^\times/\mathrm{Nr}(K_0^\times)$ is regarded as an index set. Then set $\{\gamma_j := \alpha_j^{-1}\gamma_0\alpha_j \mid j \in A\}$ forms a complete set of representatives for $\{\Gamma(b)\}$. According to (4.44), we have

$$D^\times \widehat{D}^1 \widehat{O}_F^\times = \bigsqcup_{j\in A} \big(K_0^\times \widehat{O}_F^\times\big)\alpha_j \widehat{D}^1. \tag{4.51}$$

It follows from (4.29) and (4.51) that

$$\begin{aligned}\widehat{\mathcal{E}}(\gamma_0, B) \cap D^\times \widehat{D}^1 \widehat{O}_F^\times &= \bigsqcup_{j\in A} K_0^\times \widehat{O}_F^\times \alpha_j \big(\widehat{\mathcal{E}}(\alpha_j^{-1}\gamma_0\alpha_j, B) \cap \widehat{D}^1\big) \\ &= \bigsqcup_{j\in A} K_0^\times \widehat{O}_F^\times \alpha_j \widehat{\mathcal{E}}^1(\gamma_j, B).\end{aligned} \tag{4.52}$$

For simplicity, let us write $\widehat{\mathcal{E}}^\dagger(\gamma_j, B)$ for the image of $\widehat{\mathcal{E}}^1(\gamma_j, B)$ in H, namely,

$$\widehat{\mathcal{E}}^\dagger(\gamma_j, B) := \widehat{\mathcal{E}}^1(\gamma_j, B)\widehat{O}_F^\times/\widehat{O}_F^\times \subseteq \widehat{D}^1\widehat{O}_F^\times/\widehat{O}_F^\times = \widehat{D}^\dagger.$$

Combining (4.48), (4.50) and (4.52), we get

$$\begin{aligned}&\frac{u(\mathcal{O})}{w(B)}|H_2\backslash\widehat{\mathcal{E}}^\sharp_{\mathrm{sc}}(\gamma_0, B)/U| = \operatorname{Vol}\Big(\big(\widehat{\mathcal{E}}(\gamma_0, B)\cap D^\times\widehat{D}^1\widehat{O}_F^\times\big)/\widehat{O}_F^\times,\ H_2\backslash H_1\Big) \\ &= \sum_{j\in A}\operatorname{Vol}(\alpha_j\widehat{\mathcal{E}}^\dagger(\gamma_j, B), H_2\backslash H_1) = \sum_{j\in A}\operatorname{Vol}(\widehat{\mathcal{E}}^\dagger(\gamma_j, B),\ (\alpha_j^{-1}K_0^\times\alpha_j/O_F^\times)\backslash H_1) \\ &= \sum_{\{\gamma\}\in\{\Gamma(b)\}}\operatorname{Vol}(\widehat{\mathcal{E}}^\dagger(\gamma, B),\ (K_\gamma^\times/O_F^\times)\backslash H_1),\end{aligned} \tag{4.53}$$

where $K_\gamma^\times/O_F^\times$ is equipped with the counting measure for every $\{\gamma\}\in\{\Gamma(b)\}$. Since $\widehat{\mathcal{E}}^\dagger(\gamma, B)$ is contained in the open normal subgroup $\widehat{D}^\dagger$ of H_1,

$$\operatorname{Vol}(\widehat{\mathcal{E}}^\dagger(\gamma, B),\ (K_\gamma^\times/O_F^\times)\backslash H_1) = \operatorname{Vol}(\widehat{\mathcal{E}}^\dagger(\gamma, B),\ (K_\gamma^\times\widehat{O}_F^\times)\backslash(K_\gamma^\times\widehat{O}_F^\times\widehat{D}^1)). \tag{4.54}$$

A similar proof as that of Lemma 4.6 shows that $K_\gamma^\times\widehat{O}_F^\times\cap\widehat{D}^1 = K_\gamma^1 C$. Thus we have a right $\widehat{D}^1$-equivariant bijection

$$(K_\gamma^\times\widehat{O}_F^\times)\backslash K_\gamma^\times\widehat{O}_F^\times\widehat{D}^1 \simeq (K_\gamma^1 C)\backslash\widehat{D}^1, \tag{4.55}$$

which is also measure preserving once the Haar measure on $K_\gamma^1 C$ is normalized so that $\operatorname{Vol}(C) = 1$. Indeed, the canonical images of $\widehat{\mathcal{O}}^1$ on both sides have the same volume. Thus

$$\operatorname{Vol}(\widehat{\mathcal{E}}^\dagger(\gamma, B),\ (K_\gamma^\times\widehat{O}_F^\times)\backslash K_\gamma^\times\widehat{O}_F^\times\widehat{D}^1) = \operatorname{Vol}(\widehat{\mathcal{E}}^1(\gamma, B), (K_\gamma^1 C)\backslash\widehat{D}^1).$$

Since $\widehat{\mathcal{E}}^1(\gamma, B)$ is left invariant under $K_\gamma^1 C$, we have

$$\begin{aligned}\operatorname{Vol}(\widehat{\mathcal{E}}^1(\gamma, B), (K_\gamma^1 C)\backslash\widehat{D}^1) &= \operatorname{Vol}(K_\gamma^1\backslash(K_\gamma^1 C))^{-1}\operatorname{Vol}(\widehat{\mathcal{E}}^1(\gamma, B), K_\gamma^1\backslash\widehat{D}^1) \\ &= 2\operatorname{Vol}(\widehat{\mathcal{E}}^1(\gamma, B), K_\gamma^1\backslash\widehat{D}^1).\end{aligned}$$

Combining the above calculations with (4.53), we obtain (4.32) and the proposition is proved. $\square$

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Appendix A. Polarized class sets of quaternion orders (by John Voight)

In this appendix, we give an alternate proof of the class number formula for $\mathrm{Cls}^1 \mathcal{O}$, replacing calculations with the Selberg trace formula with a direct, conceptual argument: interpreting this set as a *polarized* class set. The argument follows in the same way as the proof of the Eichler class number formula.

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A.1. **Polarized class sets.** Throughout, we follow the notation in Voight [41]. Let B be a totally definite quaternion algebra over a totally real field F; let R be the ring of integers of F and let $\mathcal{O} \subseteq B$ be an R-order.

Let I be an invertible right fractional $\mathcal{O}$-ideal such that $[\mathrm{nrd}(I)] = [R] \in \mathrm{Cl}^+ R$, i.e., the fractional R-ideal $\mathrm{nrd}(I)$ is generated by a totally positive element in F.

Definition A.1.1. A polarization of I is an element $\nu \in F^\times_{>0}$ such that $\mathrm{nrd}(I) = \nu R$. A polarized fractional right $\mathcal{O}$-ideal is a pair (I, ν) where ν is a polarization of I.

Remark A.1.2. Polarizations arise naturally in the definition of the *Shimura class group* in the theory of complex multiplication; we offer it here, by analogy in the context of quaternionic multiplication.

We define a left action of $B^\times$ on the set of polarized fractional right $\mathcal{O}$-ideals by

$$\beta \cdot (I, \nu) = (\beta I, \mathrm{nrd}(\beta)\nu). \tag{A.1.3}$$

Let $\mathrm{Cls}^1 \mathcal{O}$ be the set of equivalence classes under this action.

Lemma A.1.4. *Every class in* $\mathrm{Cls}^1 \mathcal{O}$ *is represented by a pair of the form* $(I', 1)$.

Proof. Given (I, ν), by the Hasse–Schilling theorem on norms [41, Main Theorem 14.7.4], there exists $\beta \in B^\times$ such that $\mathrm{nrd}(\beta) = \nu$. Thus $\beta^{-1}(I, \nu) = (\beta^{-1} I, 1)$. □

Using Lemma A.1.4, we deduce the following idelic description. Let $\widehat{R} := R \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ be the profinite completion of R and let $\widehat{F} := \widehat{R} \otimes_R F$ be the finite adeles of F; similarly define $\widehat{\mathcal{O}}, \widehat{B}$, etc.

Proposition A.1.5. *Suppose that* $\mathcal{O}$ *is locally norm maximal, i.e.,* $\mathrm{nrd}(\widehat{\mathcal{O}}^\times) = \widehat{R}^\times$. *Then we have a natural bijection*

$$\begin{aligned} \mathrm{Cls}^1(\mathcal{O}) &\leftrightarrow B^1 \backslash \widehat{B}^1 / \widehat{\mathcal{O}}^1 \\ [(I, 1)] &\mapsto B^1 \widehat{\alpha} \widehat{\mathcal{O}}^1 \end{aligned} \tag{A.1.6}$$

where $\widehat{I} = \widehat{\alpha}\widehat{\mathcal{O}}$ *(with* $\widehat{\alpha} \in \widehat{B}^1$*).*

Proof. First, we claim there is a natural commutative square as follows:

$$\begin{tikzcd}
\{\text{invertible right fractional } \mathcal{O}\text{-ideals } I \text{ with } \operatorname{nrd}(I) = R\} \arrow[r, leftrightarrow] \arrow[d, hook] & \widehat{B}^1/\widehat{\mathcal{O}}^1 \arrow[d, hook] \\
\{\text{invertible right fractional } \mathcal{O}\text{-ideals } I\} \arrow[r, leftrightarrow] & \widehat{B}^\times/\widehat{\mathcal{O}}^\times
\end{tikzcd} \tag{A.1.7}$$

Indeed, on the bottom row, I corresponds to $\widehat{\alpha}\widehat{\mathcal{O}}^\times$ via $\widehat{\alpha} = (\alpha_\mathfrak{p})_\mathfrak{p}$ where $I_\mathfrak{p} = \alpha_\mathfrak{p}\mathcal{O}_\mathfrak{p}$ for each prime $\mathfrak{p}$. Restricting as in the top row, since $\operatorname{nrd}(\mathcal{O}_\mathfrak{p}^\times) = R_\mathfrak{p}^\times$ we may replace $\alpha_\mathfrak{p}$ by $\alpha_\mathfrak{p}\mu_\mathfrak{p}^{-1}$ where $\mu_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}^\times$ has $\operatorname{nrd}(\alpha_\mathfrak{p}) = \operatorname{nrd}(\mu_\mathfrak{p})$, so that the local generator $\alpha_\mathfrak{p} \in B_\mathfrak{p}^1$ is now well-defined up to $\mathcal{O}_\mathfrak{p}^1$.

To finish, we consider classes to define (A.1.6). We have $[(I,1)] = [(I',1)] \in \operatorname{Cls}^1 \mathcal{O}$ if and only if $I' = \beta I$ with $\beta \in B^1$, so the map $[(I,1)] \mapsto B^1\widehat{\alpha}\widehat{\mathcal{O}}^1$ in (A.1.6) with $\widehat{\alpha} \in \widehat{B}^1$ as in the previous paragraph is well-defined. The inverse map is $B^1\widehat{\alpha}\widehat{\mathcal{O}}^1 \mapsto [(I,1)]$ where $I = \widehat{\alpha}\widehat{\mathcal{O}} \cap B$. ∎

In particular, the naturality in (A.1.6) allows us to transfer the stabilizer groups, preserving mass: the stabilizer of $[(I,1)]$ is (conjugate to)

$$\mathcal{O}_\mathsf{L}(I)^1/\{\pm 1\} = (\widehat{\alpha}\widehat{\mathcal{O}}^1\widehat{\alpha}^{-1} \cap B^1)/\{\pm 1\}, \tag{A.1.8}$$

the stabilizer of $B^1\widehat{\alpha}\widehat{\mathcal{O}}^1$.

Let $\operatorname{Cls}^{[R]} \mathcal{O} \subseteq \operatorname{Cls} \mathcal{O}$ be the subset of right fractional $\mathcal{O}$-ideal classes with $\operatorname{nrd}([I]) = [R] \in \operatorname{Cl}^+ R$. Then there is a natural forgetful (surjective) map of sets

$$\begin{aligned} \operatorname{Cls}^1(\mathcal{O}) &\to \operatorname{Cls}^{[R]}(\mathcal{O}) \\ [(I,\nu)] &\mapsto [I]. \end{aligned} \tag{A.1.9}$$

The fiber above $[I]$, after choosing a representative I, is the set of isomorphism classes of polarizations on I. This fiber is computed as follows. We have $[(I,\nu)] = [(I,\nu')] \in \operatorname{Cls}^1 \mathcal{O}$ if and only if there exists $\beta \in B^\times$ such that $\beta I = I$ and $\nu' = \operatorname{nrd}(\beta)\nu$. The former is equivalent to $\beta \in \mathcal{O}_\mathsf{L}(I)^\times$, so the fiber is a principal homogeneous space for the group $R_{>0}^\times/\operatorname{nrd}(\mathcal{O}_\mathsf{L}(I)^\times)$, a group which is well-defined independent of choice of representative.

By the naturality of (A.1.7), the map (A.1.9) gives a global interpretation for the natural idelic map

$$\begin{aligned} B^1\backslash\widehat{B}^1/\widehat{\mathcal{O}}^1 &\to B^\times\backslash B^\times\widehat{B}^1\widehat{\mathcal{O}}^\times/\widehat{\mathcal{O}}^\times \\ B^1\widehat{\alpha}\widehat{\mathcal{O}}^1 &\mapsto B^\times\widehat{\alpha}\widehat{\mathcal{O}}^\times. \end{aligned} \tag{A.1.10}$$

A.2. **Polarized class number formula.** With Proposition A.1.5 in hand, we compute the cardinality of $\operatorname{Cls}^1 \mathcal{O}$ following the same strategy that is used to compute the cardinality of $\operatorname{Cls} \mathcal{O}$, the Eichler class number formula [41, §30.8]: we convert a formula for the mass by accounting for the cardinality of stabilizers (A.1.8). More precisely, we define

$$\operatorname{mass}(\operatorname{Cls}^1 \mathcal{O}) := \sum_{[(I,1)] \in \operatorname{Cls}^1 \mathcal{O}} \frac{1}{[\mathcal{O}_\mathsf{L}(I)^1 : \{\pm 1\}]}, \tag{A.2.1}$$

which is twice of $\mathrm{Mass}^1(\mathcal{O})$ defined in (3.10). Then

$$\# \operatorname{Cls} \mathcal{O}^1 = \operatorname{mass}(\operatorname{Cls}^1 \mathcal{O}) + \sum_{[(I,1)] \in \operatorname{Cls}^1 \mathcal{O}} \left(1 - \frac{1}{[\mathcal{O}_{\mathsf{L}}(I)^1 : \{\pm 1\}]}\right). \tag{A.2.2}$$

We can express the terms in the sum in (A.2.2) as follows. For a quadratic R-algebra S in a CM-extension of F, we define the Hasse unit index by

$$Q(S) := [S^\times : S^\times_{\mathrm{tors}} R^\times] \tag{A.2.3}$$

and we let ζ_s denote a primitive sth root of unity for $s \geq 1$.

Proposition A.2.4. *We have*

$$1 - \frac{1}{[\mathcal{O}^1 : \{\pm 1\}]} = \frac{[\operatorname{nrd} \mathcal{O}^\times : R^{\times 2}]}{2} \sum_{\substack{q \geq 2 \\ [F(\zeta_{2q}):F]=2}} \left(1 - \frac{1}{q}\right) \sum_{\substack{S \subseteq F(\zeta_{2q}) \\ \# S^\times_{\mathrm{tors}} = 2q}} \frac{m(S, \mathcal{O}; \mathcal{O}^\times)}{Q(S)}.$$

Proof. We follow Voight [41, Proposition 30.8.5], counting off the elements of the group $\mathcal{O}^1/\{\pm 1\}$ by maximal cyclic subgroups of some order $q \geq 2$. By essentially the same argument, we obtain

$$[\mathcal{O}^1 : \{\pm 1\}] - 1 = \sum_{\substack{q \geq 2 \\ [F(\zeta_{2q}):F]=2}} \sum_{\substack{S \subseteq F(\zeta_{2q}) \\ \# S^\times_{\mathrm{tors}} = 2q}} (q-1) m(S, \mathcal{O}; \mathcal{O}^\times) \frac{[\mathcal{O}^\times : R^\times]}{2[S^\times : R^\times]}$$

where $m(S, \mathcal{O}; \mathcal{O}^\times)$ counts the number of optimal embeddings $\phi \colon S \hookrightarrow \mathcal{O}$ up to conjugation by $\mathcal{O}^\times$.

We then make two substitutions to simplify. First, from the exact sequence

$$1 \to S^\times_{\mathrm{tors}}/R^\times_{\mathrm{tors}} \to S^\times/R^\times \to S^\times/(S^\times_{\mathrm{tors}} R^\times) \to 1 \tag{A.2.5}$$

and $R^\times_{\mathrm{tors}} = \{\pm 1\}$, we have

$$[S^\times : R^\times] = Q(S) q. \tag{A.2.6}$$

Second, from the exact sequence

$$1 \to \frac{\mathcal{O}^1}{\{\pm 1\}} \to \frac{\mathcal{O}^\times}{R^\times} \xrightarrow{\operatorname{nrd}} \frac{\operatorname{nrd}(\mathcal{O}^\times)}{R^{\times 2}} \to 1 \tag{A.2.7}$$

we get

$$[\mathcal{O}^\times : R^\times] = [\operatorname{nrd} \mathcal{O}^\times : R^{\times 2}][\mathcal{O}^1 : \{\pm 1\}]. \tag{A.2.8}$$

Substituting these in and dividing by $[\mathcal{O}^1 : \{\pm 1\}]$ gives the result. □

We now finish the polarized class number formula for $\mathrm{Cls}^1(\mathcal{O})$ following the same arguments as those for [41, Theorem 30.8.6], recalling the setup for selectivity [41, §31.1]. The spinor genus $\operatorname{SpnGen} \mathcal{O} \subseteq \operatorname{Gen} \mathcal{O}$ of $\mathcal{O}$ is the set of R-orders $\mathcal{O}' \subseteq B$ such that $\widehat{\mathcal{O}}' = \widehat{\alpha}^{-1} \widehat{\mathcal{O}} \widehat{\alpha}$ for some $\widehat{\alpha} \in B^\times \widehat{B}^1 \leq \widehat{B}^\times$.

Definition A.2.9. We say $\operatorname{Gen} \mathcal{O}$ is spinor genial for S if one of the following conditions holds:

- (i) $\operatorname{Emb}(S, \mathcal{O}') = \emptyset$ for every order $\mathcal{O}' \in \operatorname{Gen} \mathcal{O}$; or
- (ii) every spinor genus $\operatorname{SpnGen} \mathcal{O}' \subseteq \operatorname{Gen} \mathcal{O}$ contains at least one order $\mathcal{O}'' \in \operatorname{SpnGen} \mathcal{O}'$ with $\operatorname{Emb}(S, \mathcal{O}'') \neq \emptyset$.

If $\operatorname{Gen}\mathcal{O}$ is not spinor genial for S, then we say that it is **spinor optimally selective** for S.

If $\operatorname{Gen}\mathcal{O}$ is spinor optimally selective for S, then for each spinor genus $\operatorname{SpnGen}\mathcal{O}' \subseteq \operatorname{Gen}\mathcal{O}$, then exactly one of the following holds:

- if $\operatorname{Emb}(S,\mathcal{O}'') \neq \emptyset$ for some $\mathcal{O}'' \in \operatorname{SpnGen}\mathcal{O}'$, then we say $\operatorname{SpnGen}\mathcal{O}'$ **is selected by** S;
- otherwise, $\operatorname{Emb}(S,\mathcal{O}'') = \emptyset$ for every order $\mathcal{O}'' \in \operatorname{SpnGen}\mathcal{O}'$, and we say $\operatorname{SpnGen}\mathcal{O}'$ is **not selected by** S.

By strong approximation, condition (i) of Definition A.2.9 is equivalent to the existence of a prime $\mathfrak{p}$ of F such that $\operatorname{Emb}(S_\mathfrak{p},\mathcal{O}_\mathfrak{p}) = \emptyset$.

Theorem A.2.10. *Suppose that $\mathcal{O}$ is residually unramified. Then*

$$\#\operatorname{Cls}^1\mathcal{O} = \operatorname{mass}(\operatorname{Cls}^1\mathcal{O}) + \frac{1}{2h(R)}\sum_{q\geq 2}\left(1-\frac{1}{q}\right)\sum_{\substack{S\\ \#S^\times_{\mathrm{tors}}=2q}}\delta(S,\mathcal{O})\frac{h(S)}{Q(S)}m(\widehat{S},\widehat{\mathcal{O}};\widehat{\mathcal{O}}^\times)$$

where

$$\delta(S,\mathcal{O}) := \begin{cases} 1, & \textit{if } \operatorname{Gen}\mathcal{O} \textit{ is spinor genial for } S; \\ 2, & \textit{if } \operatorname{Gen}\mathcal{O} \textit{ is spinor optimally selective for } S \textit{ and} \\ & \operatorname{SpnGen}\mathcal{O} \textit{ is selected by } S; \\ 0, & \textit{if } \operatorname{Gen}\mathcal{O} \textit{ is spinor optimally selective for } S \textit{ and} \\ & \operatorname{SpnGen}\mathcal{O} \textit{ is not selected by } S. \end{cases}$$

Proof. Let

$$w^1_{(I,\nu)} = w^1_I := \#\mathcal{O}_{\mathsf{L}}(I)^1/\{\pm 1\}.$$

We begin with

$$\#\operatorname{Cls}^1\mathcal{O} - \operatorname{mass}(\operatorname{Cls}^1\mathcal{O}) = \sum_{[(I,\nu)]\in\operatorname{Cls}^1\mathcal{O}}\left(1-\frac{1}{w^1_{(I,\nu)}}\right). \tag{A.2.11}$$

From the fiber count in (A.1.9), then substituting Proposition A.2.4, we obtain

(A.2.12)
$$\begin{aligned}\sum_{[(I,\nu)]\in\operatorname{Cls}^1\mathcal{O}}\left(1-\frac{1}{w^1_{(I,\nu)}}\right) &= \sum_{[I]\in\operatorname{Cls}^{[R]}\mathcal{O}}[R^\times_{>0} : \operatorname{nrd}(\mathcal{O}_{\mathsf{L}}(I)^\times)]\left(1-\frac{1}{w^1_I}\right) \\ &= \frac{[R^\times_{>0} : R^{\times 2}]}{2}\sum_{q\geq 2}\left(1-\frac{1}{q}\right)\sum_{\substack{S\\ S^\times_{\mathrm{tors}}=2q}}\sum_{[I]\in\operatorname{Cls}^{[R]}\mathcal{O}}\frac{m(S,\mathcal{O}_{\mathsf{L}}(I);\mathcal{O}_{\mathsf{L}}(I)^\times)}{Q(S)}.\end{aligned}$$

We finish by substituting in the spinor trace formula (Proposition 2.8), using the fact that $\#\operatorname{Cl}^+ R = [R^\times_{>0} : R^{\times 2}](\#\operatorname{Cl} R)$. ∎

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