ERRATA: ON THE PARAMODULARITY OF TYPICAL ABELIAN SURFACES (AND NEW APPENDIX: REDUCTION OF G-COVARIANT BILINEAR FORMS)

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(APPENDIX BY J.-P. SERRE)

Abstract. We note a mathematical error in the article "On the paramodularity of typical abelian surfaces" [Algebra Number Theory **13** (2019), no. 5, 1145–1195]; we correct it using a result of Serre, which he proves in an appendix. Serre's result extends his work on the reduction of G-invariant bilinear forms modulo primes to the case of G-covariant forms.

This note gives errata for the article *On the paramodularity of typical abelian surfaces* [BPPTVY]. In the appendix, Serre proves a result of independent interest, generalizing his previous results [S] to the covariant case (see the introduction below).

(1) Definition 2.1.2: should be "if the values $\operatorname{tr}\rho(\mathrm{Frob}_{\mathfrak{p}})$ belong to a computable subring" (error pointed out by Minhyong Kim). In general, $\operatorname{tr}\rho$ may take other values in $\mathbb{Z}_\ell$ for arbitrary elements $\sigma \in \mathrm{Gal}_{F,S}$, but all that is accessed are the values $\operatorname{tr}\rho(\mathrm{Frob}_{\mathfrak{p}})$.

(2) In §4.2, the group $\mathrm{GSp}_4^+(\mathbb{R})$ was only defined implicitly. Explicitly,

$$\mathrm{GSp}_4^+(\mathbb{R}) := \{M \in \mathrm{GL}_4(\mathbb{R}) : M^{\mathsf{T}} J M = \mu J \text{ for some } \mu \in \mathbb{R}_{>0}\}.$$

(3) In §5, we worked seemingly interchangeably with $\mathrm{GSp}_4(\mathbb{F}_2)$ and $\mathrm{Sp}_4(\mathbb{F}_2)$, but we neglected to note that these groups are equal $\mathrm{GSp}_4(\mathbb{F}_2) = \mathrm{Sp}_4(\mathbb{F}_2)$ (any similitude factor belongs to $\mathbb{F}_2^\times$ so is necessarily trivial).

(4) We are grateful to J.-P. Serre for pointing out an error in our paper and providing a correction. In the proof of our Lemma 4.3.6, we mistakenly applied a result of Serre [S, Theorem 5.1.4]: to transform a covariant bilinear form (having nontrivial similitude character) into an invariant bilinear form, we modified the involution $\sigma \mapsto \sigma^{-1}$ to $\sigma \mapsto \sigma^* := \epsilon(\sigma)\sigma^{-1}$. However, this map $\sigma \mapsto \sigma^*$ is no longer an involution! To correct this error, Serre has extended his result to the case of covariant bilinear forms, so our appeal to his result is now direct (Theorem 1 below); and he has allowed us to include it in the following appendix.

(5) (5.3.2): 2^e should be 2^k.

(6) The reference [53] (Jean-Pierre Serre, *Résumé des cours de 1984–1985*, Annuaire du Collège de France 1985, 85–90) is more conveniently found at:

Jean-Pierre Serre, *Oeuvres/Collected papers IV (1985–1998)*, Springer Collected Works in Math., Springer, Heidelberg, 2000, no. 135, 27–32.

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Appendix: Reduction of G-covariant bilinear forms, by J.-P. Serre

Introduction. This note is intended as a complement to [S] where reductions of G-invariant bilinear forms modulo primes were studied. Indeed, in most applications to ℓ-adic representations the natural bilinear forms are not G-invariant; they are only covariant with respect to a character of the group G. The simplest example of this is the $\mathbb{Q}_\ell$-Tate module V_ℓ of an abelian variety A over a field F of characteristic $\neq \ell$: a polarization of A defines a nondegenerate alternating form B on V_ℓ, which is covariant under the action of the absolute Galois group $\Gamma_F = \mathrm{Gal}(F_s/F)$, namely:

$$B(gx, gy) = \chi_\ell(g) B(x, y) \ \text{ for every } g \in \Gamma_F, x, y \in V_\ell,$$

where χ_ℓ is the ℓ-cyclotomic character.

We shall see that the results of [S] extend to the covariant case, with practically the same proofs.

1. The setting. It is almost the same as that of [S]. Namely:

G is a group,

K is a field with a discrete valuation,

R is the ring of integers of K,

π is a uniformizer of K,

$k = R/\pi R$ is the residue field,

$\varepsilon \colon G \to R^\times$ is a homomorphism,

V is a finite dimension K-vector space on which G acts, in such a way that there exists an R-lattice of V which is G-stable ("bounded action"),

V_k is the k-vector space obtained by the semisimplification of the $k[G]$-module $L/\pi L$, where L is a G-stable lattice of V; up to isomorphism, it is independent from the choice of L,

B is a symmetric (resp. alternating) nondegenerate K-bilinear form on V, which is ε-covariant under the action of G, i.e.

$$B(gx, gy) = \varepsilon(g) B(x, y) \ \text{ for } g \in G, x, y \in V. \tag{1.1}$$

2. Statement of the theorems. The main theorem is the analogue of Theorem A of [S]. Namely:

Theorem 1. *There exists a nondegenerate symmetric (resp. alternating) k-bilinear form on V_k such that*

$$b(gx, gy) = \varepsilon(g) b(x, y) \ \text{ for } g \in G, x, y \in V_k. \tag{1.2}$$

As in [S], the proof will use the following complement to a classical theorem of Brauer and Nesbitt:

Theorem 2. *Let E be a finite dimensional $k[G]$-module endowed with a nondegenerate symmetric (resp. alternating) k-bilinear form b having property (1.2). Then, the semisimplification E^{ss} of E has a k-bilinear form with the same properties as b.*

3. Proof of theorem 2. Use induction on $\dim E$. Assume $E \neq 0$ and choose a minimal nonzero G-submodule S of E. Let $H \subset E$ be the orthogonal subspace of S with respect to b. Since S is minimal, there are two possibilities:

a) $H \cap S = 0$, i.e. the restriction of b to S is nondegenerate. In that case, we have $E^{\mathrm{ss}} = S \oplus H^{\mathrm{ss}}$ and we apply the induction hypothesis to H.

b) $H \cap S = S$, i.e. S is totally isotropic for b. We have
$E^{\mathrm{ss}} = (S \oplus E/H) \oplus (H/S)^{\mathrm{ss}}$.

The induction hypothesis applies to $(H/S)^{\mathrm{ss}}$. As for the first factor $S \oplus E/H$, one defines a bilinear form $b_1(x, y)$ on it by the following rule: if x, y both belong to S, or to E/H, then $b_1(x, y) = 0$; if $x \in S$ and $y \in E/H$, then $b_1(x, y) = b(x, y')$ where y' is any representative of y in E; if $x \in E/H$ and $y \in S$, then $b_1(x, y) = b_1(y, x)$ in the symmetric case and $b_1(x, y) = -b_1(y, x)$ in the alternating case. It is clear that the form b_1 has the required properties.

4. Proof of Theorem 1. The first step ([S, Theorem 5.2.1]) is to show the existence of a lattice L in V, which is G-stable, and almost self-dual, i.e. $\pi L' \subset L \subset L'$, where L' is the dual of L (note that formula (1.1) implies that the dual of a G-stable lattice is G-stable). This is done by choosing a G-stable lattice M, and defining L as the "lower middle" $m_-(M, M')$ of M and its dual M' :

$m_-(M, M') =$ smallest lattice containing $\pi^n M \cap \pi^{-n} M'$ for every $n \in \mathbb{Z}$.

It is proved in [S, Theorem 3.1.1] that $m_-(M, M')$ is an almost self-dual lattice.

The second step is to define a bilinear form b on the k-vector space $E = L/\pi L' \oplus L'/L$ by using the reduction mod π of B on $L/\pi L'$, and of πB on L'/L. It is clear that b is nondegenerate, ε-covariant, and symmetric (resp. alternating) if B is. By Theorem 2, the semisimplification E^{ss} of E has a bilinear form with the required properties. Since E^{ss} is isomorphic to V_k, this proves Theorem 1.

References

[BPPTVY] Armand Brumer, Ariel Pacetti, Cris Poor, Gonzalo Tornaría, John Voight, and David S. Yuen, *On the paramodularity of typical abelian surfaces*, Algebra & Number Theory **13** (2019), no. 5, 1145–1195.

[S] Jean-Pierre Serre, *On the mod p reduction of orthogonal representations*, Lie groups, geometry, and representation theory, eds. Victor G. Kac and Vladimir L. Popov, Progr. Math., vol. 326, Birkhäuser, 2018, 527–540.

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