MATH 255: ELEMENTARY NUMBER THEORY
HOMEWORK #2

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The problems with an asterisk * are optional (as they are potentially challenging!). If you attempt them, they will be graded out of 5 and added to your homework score.

§3.1: Prime Numbers

Problem 3.1.1. Determine which of the following integers are primes.

(a) 101
(b) 103
(c) 107

Problem 3.1.7. Show that if a and n are positive integers with $n > 1$ and $a^n - 1$ is prime, then $a = 2$ and n is prime. *[Hint: Use the identity $a^{kl} - 1 = (a^k - 1)(a^{k(l-1)} + a^{k(l-2)} + \cdots + a^k + 1)$.]*

Problem 3.1.8. [This exercise constructs another proof of the infinitude of primes.] Show that the integer $Q_n = n! + 1$, where n is a positive integer, has a prime divisor greater than n. Conclude that there are infinitely many primes.

Problem 3.1.11. Let $Q_n = p_1 p_2 \cdots p_n + 1$, where $p_1, p_2, \ldots, p_n$ are the n smallest primes. Determine the smallest prime factor of Q_n for $n = 1, 2, 3, 4, 5, 6$. Do you think that Q_n is prime infinitely often? *[Note: This is an unresolved question.]*

Problem 3.1.23*. Show that if $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where the coefficients are integers, then there is an integer y such that $f(y)$ is composite. *[Hint: Assume that $f(x) = p$ is prime, and show that p divides $f(x + kp)$ for all integers k. Conclude that there is an integer y such that $f(y)$ is composite from the fact that a polynomial of degree n, $n > 1$, takes on each value at most n times.]*

§3.2: The Distribution of Primes

Problem 3.2.2. Find one million consecutive composite integers.

Problem 3.2.3. Show that there are no "prime triplets", that is, primes p, $p+2$, and $p+4$, other than $3, 5, 7$.

Problem 3.2.10. Verify Goldbach's conjecture for each of the following values of n.

(a) 50
(c) 102
(e) 200

Date: Due Wednesday, 28 January 2009.

Problem 3.2.12. Show that every integer greater than 11 is the sum of two composite integers.

Problem 3.2.A. Show that $x^2 + 3x - \log x \sim x^2$.

Computation 3.2.3*. Verify Goldbach's conjecture for all even positive integers less than 10000.