MATH 252: ABSTRACT ALGEBRA II
HOMEWORK #1

Problem 1.

(a) Let $a = 3 - 8i$ and $b = 2 + 3i$. Find $x, y \in \mathbb{Z}[i]$ such that $ax + by = 1$.
(b) Show explicitly that the ideal $I = (85, 1 + 13i) \subset \mathbb{Z}[i]$ is principal by exhibiting a generator.

Problem 2 (DF 8.1.6). The following problem is colloquially known as the *postage stamp problem.* For bureaucratic reasons, the Postal Service decides that it will now print only two stamps. Outraged, an angry mob insists that this will make mailing packages impossible.

Show that the angry mob is wrong. Let $a, b \in \mathbb{Z}_{>0}$ be relatively prime. We say that $N \in \mathbb{Z}$ is a *linear combination* of a, b if there exists $x, y \in \mathbb{Z}$ such that $N = ax + by$, and we say that N is a *nonnegative* linear combination if $x, y \in \mathbb{Z}_{\geq 0}$. Show that $ab - a - b$ cannot be written as a nonnegative linear combination of a, b, but every $N > ab - a - b$ can. Conclude that every sufficiently large postage can be obtained with only stamps with denominations a and b.

Problem 3 (sorta DF 8.1.8). Show that the ring $\mathbb{Z}[\rho]$ where $\rho = (-1 + \sqrt{-3})/2$ is a Euclidean domain. *[Hint: Plot the points of $\mathbb{Z}[\rho]$ in $\mathbb{C}$ and mimic the proof for $\mathbb{Z}[i]$.]*

Problem 4 (DF 8.1.3). Let R be a Euclidean domain. Let m be the smallest (nonnegative) integer in the set $\{N(a) : 0 \neq a \in R\}$.

(a) Prove that every nonzero element of R of norm m is a unit.
(b) Deduce that a nonzero element of norm zero is a unit, and show by example that the converse of this statement is false.

Problem 5 (sorta DF 8.2.5). Let $R = \mathbb{Z}[\sqrt{-5}]$.

(a) Show that the ideal $(2, 1 + \sqrt{-5})$ is not a principal ideal.
(b) Let $I = (3, 2 + \sqrt{-5})$ and $J = (3, 2 - \sqrt{-5})$. Show that the product

$$IJ = \{\textstyle\sum_i x_i y_i : x_i \in I, y_i \in J\} = (3)$$

is principal.

Problem 6. Factor the element 390 into irreducibles in $\mathbb{Z}[i]$. *[Hint: See* Proposition 18 *in §8.3.]*

Problem 7 (sorta DF 9.1.14). Let R be an integral domain, and let $S = R[x, y]$.

(a) Prove that the ideal $(x^4 - y^2)$ is not a prime ideal in S.
(b) Prove that the ideal $(x^3 - y^2)$ is a prime ideal in S. *[Hint: Consider the ring homomorphism $\phi : R[x, y] \to R[t]$ with $x, y \mapsto t^2, t^3$.]*

Problem 8. Let $R = \mathbb{F}_3[x]/(x^3 - x - 1)$.

(a) Show that $\#R = 27$. *[Hint: Use the division algorithm.]*
(b) Prove that R is a field.
(c) Conclude that $R^\times$ has an element of order 13; in fact, show that x is such an element.

Problem 9 (sorta DF 9.4.20). Here we see some pathologies in $R[x]$ when R is not an integral domain.

(a) Show that in $\mathbb{Z}/6\mathbb{Z}[x]$, the polynomial x factors as $x = (3x + 4)(4x + 3)$.
(b) Show that the ideal $(3, x)$ is a principal ideal in $\mathbb{Z}/6\mathbb{Z}[x]$.

Date: 14 January 2008; due Friday, 1 February 2008.