MATH 250B: COMMUTATIVE ALGEBRA
HOMEWORK #1

JOHN VOIGHT

Some solutions are omitted or sketched.

Problem 4. *Let $A_1, \dots, A_r$ be row vectors of dimension n over a field k. Let $X = (x_1, \dots, x_n)$. Let $b_1, \dots, b_r \in k$. By a* system of linear equations *in k one means a system of type*

$$A_1 \cdot X = b_1, \ \dots, \ A_r \cdot X = b_r.$$

If $b_1 = \dots = b_r = 0$, one says the system is homogeneous. *We call n the number of variables, and r the number of equations. A solution X of the homogeneous system is called* trivial *if $x_i = 0$, $i = 1, \dots, n$.*

(a) *Show that a homogeneous system of r linear equations in n unknowns with $n > r$ always has a nontrivial solution.*
(b) *Let L be a system of homogeneous linear equations over a field k. Let k be a subfield of k'. If L has a non-trivial solution in k', show that it has a non-trivial solution in k.*

Solution. The matrix A with rows A_i defines a map $L_A : k^n \to k^r$ by $X \mapsto A \cdot X$. By SS2, if $W = \ker L_A$, then $\dim W + \operatorname{rk} A = n$. Since A has only r rows, it has (row) rank $\leq r$, so $\dim W \geq n - r > 0$, so the system has a nontrivial solution. This proves (a).

For (b), let $x = (x_1, \dots, x_n) \in (k')^n$ be a solution to $Ax = 0$. Consider k' as a vector space over k, and let V be the k-subvector space of k' generated by the x_i. (Note that even if k' is infinite-dimensional over k, V is necessarily finite dimensional.) Let $v_1, \dots, v_m$ be any basis of V and write $x_i = \sum_j c_{ij} v_j$.

We claim that for each j, $c_j = (c_{ij})_i \in k^n$ is a solution; since the x_i were not all zero, at least one c_j is not zero. The claim follows from the calculation:

$$Ax = A \begin{pmatrix} \sum_j c_{1j} v_j \\ \vdots \\ \sum_j c_{nj} v_j \end{pmatrix} = A \begin{pmatrix} c_{11} \\ \vdots \\ c_{n1} \end{pmatrix} v_1 + \dots + A \begin{pmatrix} c_{1m} \\ \vdots \\ c_{nm} \end{pmatrix} v_m = (Ac_1)v_1 + \dots + (Ac_m)v_m.$$

Since the v_i are linearly independent, $Ac_j = 0$ for all j.

Problem 5. *Let M be an $n \times n$ matrix over a field k. Assume that* $\operatorname{tr}(MX) = 0$ *for all $n \times n$ matrices X in k. Show that $M = O$.*

Date: January 30, 2003.
XIII: 4–6, 8–9, 11–18.

Solution. Denote by E_{ij} the matrix whose only nonzero entry is $e_{ij} = 1$, and let $M = (m_{ij})_{i,j}$. Then since

$$\operatorname{tr}(MX) = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} x_{ji}$$

we see that $ME_{ij} = m_{ji}$. Hence if $\operatorname{tr}(ME_{ij}) = 0$ for all i, j, then $M = O$.

Problem 6. *Let S be a set of $n \times n$ matrices over a field k. Show that there exists a column vector $X \neq 0$ of dimension n in k such that $MX = X$ for all $M \in S$ if and only if there exists such a vector in some extension field k' of k.*

Solution. Since $k \subset k'$, one direction is trivial. For the other, note that if $MX' = X'$ for all $M \in S$ and $X' \in (k')^n$, then $(M - I)X' = 0$ for all $M \in S$. By the solution to 4(b), choosing a basis for the components for X' over k, we obtain an element $X \in k^n$ such that $(M - I)X = 0$ for all $M \in S$.

Problem 8. *Let N be a strictly upper triangular $n \times n$ matrix, that is $N = (a_{ij})$ and $a_{ij} = 0$ if $i \geq j$. Show that $N^n = 0$.*

Solution. Treating T as an indeterminant, note that $\det(N - TI) = (-T)^n$, since $N - TI$ is (not strictly) upper triangular with $-T$ along the diagonal. By the Cayley-Hamilton theorem, N satisfies its characteristic polynomial $T^n = 0$, so $N^n = 0$.

Problem 9. *Let E be a vector space over k of dimension n. Let $T : E \to E$ be a linear map such that T is nilpotent, that is $T^m = 0$ for some positive integer m. Show that there exists a basis of E over k such that the matrix of T with respect to this basis is strictly upper triangular.*

Solution. We construct such a basis inductively. Since T is nilpotent, there exists a v_1 such that $Tv_1 = 0$ (e.g. since $(\det T)^m = \det T^m = 0$).

Now for $i = 1, \ldots, n - 1$, given linearly independent vectors $v_1, \ldots, v_i$, we claim there exists a vector v_{i+1} independent of the v_j such that $Tv_{i+1} \in \operatorname{span}\{v_1, \ldots, v_i\}$. Otherwise, $Tv \notin \operatorname{span}\{v_j\}$ for all $v \notin \operatorname{span}\{v_j\}$, and this contradicts $T^m v = 0$.

This gives a basis $v_1, \ldots, v_n$ of E. In this basis, the ith column of T has $t_{ij} = 0$ for $i \geq j$ since $Tv_{i+1} \in \operatorname{span}\{v_1, \ldots, v_i\}$, so T is strictly upper triangular.

Problem 11. *Let R be the set of all upper triangular $n \times n$ matrices (a_{ij}) with a_{ij} in some field k, so $a_{ij} = 0$ if $i > j$. Let J be the set of all strictly upper triangular matrices. Show that J is a two-sided ideal in R. How would you describe the factor ring R/J?*

Solution. Clearly, J is an additive subgroup of R. If $M = (m_{ij}) \in J$ and $A = (a_{ij}) \in R$, writing $AM = B$ we see that

$$b_{ij} = \sum_{k} a_{ik} m_{kj}.$$

For $i \geq j$, $a_{i1} = \cdots = a_{i(i-1)} = 0$ while $m_{ij} = m_{(i+1)j} = \cdots = m_{nj} = 0$, so $b_{ij} = 0$. A similar result holds for MA, so J is a two-sided ideal.

Now $R/J \cong k^n$ as rings. Clearly, any matrix $A \in R$ can be written $A = D + M$ where D is a diagonal matrix and $M \in J$; the ring of diagonal matrices is isomorphic to k^n because addition and multiplication are componentwise.

Problem 14. *Let F be any field. Let D be the subgroup of diagonal matrices in $GL_n(F)$. Let N be the normalizer of D in $GL_n(F)$. Show that N/D is isomorphic to the symmetric group on n elements.*

Solution (sketch). The group N is the group of matrices with exactly one nonzero element in each row and column. Every such matrix can be written as a permutation matrix (a matrix of zeros and ones contained in N) and a diagonal matrix, so $N/D \cong S_n$ by the action of the permutation matrix on the standard basis vectors.

Problem 16. *Let F be a finite field with q elements. Show that the order of $SL_n(F)$ is*

$$q^{n(n-1)/2} \prod_{i=2} n(q^i - 1)$$

and that the order of $PSL_n(F)$ is

$$\frac{1}{d} q^{n(n-1)/2} \prod_{i=2} n(q^i - 1).$$

Solution. The determinant map $\det : GL_n(F) \to F^*$ is clearly surjective, and has kernel $SL_n(F)$, so $SL_n(F) \cong GL_n(F)/F^*$. Hence $\#SL_n(F) = \#GL_n(F)/(q-1)$, which is the result.

We know that $PSL_n(F) = SL_n(F)/Z$, where Z is the group of scalar matrices which are nth roots of unity (see after Theorem 9.2). Since F^* is cyclic, $\#\mu_n(F^*) = \gcd(n, q-1) = d$. This gives the second statement.

Problem 17. *Let F be a finite field with q elements. Show that the group of all upper triangular matrices with* 1 *along the diagonal is a Sylow subgroup of $GL_n(F)$ and $SL_n(F)$.*

Solution. We know from Exercise 12 that this is a subgroup. It has order $q^{n(n-1)/2}$, since we can choose the strictly upper triangular elements arbitrarily from F. If $q = p^m$, then we see it is a p-subgroup; by Exercise 15, this is the largest power of p that divides $\#GL_n(F)$ (and $\#SL_n(F)$), so it is a p-Sylow subgroup as well.