MATH 052: FUNDAMENTALS OF MATHEMATICS
FINAL EXAM

Name ______________________________________

Problem	**Score**	**Problem**	**Score**
1		6	
2		7	
3		8	
4		9	
5		10	

Total ____________

Date: 14 December 2012.

Problem 1.

(a) List all elements of the set $\{k \in \mathbb{Z} : k \text{ is prime and } 1 < k^3 \leq 200\}$.

(b) Label as true or false:

$$\emptyset \subseteq \{s, t\} \text{ and } \emptyset \in \{\{\emptyset\}\}.$$

(c) Let $A = \{x \in \mathbb{Z} : 5 \mid x\}$ and let $B = \{x \in \mathbb{Z} : 3 \mid x\}$. Find $(\overline{A} \cup B) \cap [2, 16]$, where $\overline{A}$ denotes the complement of A in $\mathbb{Z}$.

(d) Let $A = \{1\}$ and $C = \{1, 2\}$. Give an example of a set B such that

$$\mathcal{P}(A) \subsetneq B \subsetneq \mathcal{P}(C),$$

where $\mathcal{P}$ denotes the power set.

Problem 2. Use a truth table to show that

$$((P \wedge Q) \Rightarrow R) \equiv ((P \wedge (\sim R)) \Rightarrow (\sim Q)).$$

Problem 3. Consider the function

$$\begin{aligned} f : \mathbb{R} \setminus \{1\} &\to \mathbb{R} \setminus \{0\} \\ x &\mapsto f(x) = \frac{2}{x-1} \end{aligned}$$

Show that f is a bijection and find the inverse of f.

Problem 4.

(a) Compute $d = \gcd(172, 39)$ using the Euclidean algorithm.

(b) Compute $x, y \in \mathbb{Z}$ such that $172x + 39y = d$.

Problem 5. Let A, B be sets. Prove (rigorously) de Morgan's law:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

Problem 6. Prove by induction that for all integers $n \geq 1$

$$\frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Problem 7. Let $a, b \in \mathbb{Z}$. Prove the statement

$$\text{If } ab \text{ is odd, then } a^2 + b^2 \text{ is even}$$

in two steps.

(a) First, prove that if ab is odd then a, b are both odd.

(b) Use (a) to complete the proof.

Problem 8.

(a) Prove that $\sqrt{2}+1$ is irrational. (You may use the fact that $\sqrt{2}$ is irrational.)

(b) Express the following statement in symbols and show that it is a true statement:
There exists an integer n such that $x^2+y^2 \geq n$ for every two real numbers x and y.

(c) Compute $[-3]\cdot[13]=[r]$ in $\mathbb{Z}/11\mathbb{Z}$ with $0 \leq r < 11$.

Problem 9.

(a) How many partitions of $\{-6, 0, 13\}$ are there?

(b) Let R be an equivalence relation on $A = \{a, b, c, d, e, f, g\}$ such that

$$aRc,\ cRd,\ dRg,\ \text{and } bRf.$$

Suppose there are three distinct equivalence classes for R. Determine these equivalence classes.

(c) A relation R is defined on $\mathbb{Z}$ by aRb if $3 \mid (a^3 - b)$. Is R reflexive? If so, give a proof; if not, give a disproof.

Problem 10. Let $f : A \to B$ and $g : B \to C$ be functions. Show that if f and g are bijective, then $g \circ f$ is bijective.