MATH 052: FUNDAMENTALS OF MATHEMATICS
FINAL EXAM SOLUTIONS

Problem 1. For (a), we have $\{2, 3, 5\}$, since $7^3 > 200$. Statement (b) is false, since $\emptyset \notin \{\{\emptyset\}\}$. For (c), we have $(A^c \cup B) \cap [2, 16] = \{x \in \mathbb{Z} : 5 \nmid x \text{ or } 3 \mid x \text{ and } 2 \le x \le 16\} = \{2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16\}$. For (d), we $\mathcal{P}(A) = \{\emptyset, \{1\}\}$ and $\mathcal{P}(C) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, so we could take $B = \{\emptyset, \{1\}, \{2\}\}$ or $B = \{\emptyset, \{1\}, \{1, 2\}\}$.

Problem 2. For the truth table, we have:

P	Q	R	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$	$P \wedge (\sim R)$	$(P \wedge (\sim R)) \Rightarrow (\sim Q)$
F	F	F	F	T	F	T
F	F	T	F	T	F	T
F	T	F	F	T	F	T
F	T	T	F	T	F	T
T	F	F	F	T	T	T
T	F	T	F	T	F	T
T	T	F	T	F	T	F
T	T	T	T	T	F	T

Since the fifth and seventh columns have the same truth values, they are logically equivalent.

Problem 3. First we show injectivity: if $x_1, x_2 \in \mathbb{R} \setminus \{1\}$ and $f(x_1) = f(x_2)$, then $2/(x_1 - 1) = 2/(x_2 - 1)$ so cross-multiplying we get $2(x_1 - 1) = 2(x_2 - 1)$ and simplifying gives $x_1 = x_2$, so f is injective. To show surjectivity, we take $y = f(x) = 2/(x - 1)$ and solve for x to obtain $x = 1 + 2/y$: then given $y \in \mathbb{R} \setminus \{0\}$ we let $x = 1 + 2/y \in \mathbb{R} \setminus \{1\}$ and indeed

$$f(1 + 2/y) = \frac{2}{(1 + 2/y) - 1} = \frac{2}{2/y} = y.$$

So f is surjective. This also shows that $f^{-1}(y) = 1 + 2/y$.

Problem 4. For (a), we have

$$\begin{aligned} 172 &= 4 \cdot 39 + 16 \\ 39 &= 2 \cdot 16 + 7 \\ 16 &= 2 \cdot 7 + 2 \\ 7 &= 3 \cdot 2 + 1. \end{aligned}$$

Thus $\gcd(315, 271) = 1$.

For (b), we have

$$1 = 7 + (-3) \cdot 2 = (-3) \cdot 16 + 7 \cdot 7 = 7 \cdot 39 + (-17)(16) = (-17)(172) + (75)(39).$$

So $x = -17$ and $y = 75$.

Problem 5. First we show $(\subseteq)$. Let $x \in \overline{A \cup B}$. Then $x \notin A \cup B$, so ("it is not the case that $x \in A$ or $x \in B$") by de Morgan's law in logic, $x \notin A$ and $x \notin B$. Thus $x \in \overline{A}$ and $x \in \overline{B}$ so $x \in \overline{A} \cap \overline{B}$.

Now we show $(\supseteq)$. Let $x \in \overline{A} \cap \overline{B}$. Then $x \in \overline{A}$ and $x \in \overline{B}$ and $x \notin A$ and $x \notin B$. By de Morgan's law, we have $x \notin A \cup B$, so $x \in \overline{A \cup B}$.

Thus $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Problem 6. We proceed by induction. The base case $n = 1$ is the equality $1/2 = 1/(1 + 1)$, which is true.

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Now suppose that the statement is true for $n = k$:

$$\frac{1}{2} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}.$$

We want to show the statement is true for $n = k + 1$:

$$\frac{1}{2} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

By the inductive hypothesis,

$$\begin{aligned} \frac{1}{2} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}. \end{aligned}$$

By the principle of mathematical induction, the statement is true for all n.

Problem 7. First, (a): if ab is odd, then a and b are both odd. To prove this, we prove the contrapositive: if a is even or b is even, then ab is even. Indeed, if a is even or b is even, then without loss of generality, $a = 2x$ is even, and then $ab = (2x)b = 2(bx)$ is even by definition, since $bx \in \mathbb{Z}$.

Now we use the claim to prove the result (b). Suppose that ab is odd; then by the claim, a and b are both odd. Thus $a = 2s + 1$ and $b = 2t + 1$. Thus

$$a^2 + b^2 = (2s+1)^2 + (2t+1)^2 = 2(2s^2 + 2s + 2t^2 + 2t + 1)$$

is even.

Problem 8. For (a), we proceed by contradiction. Suppose that $\sqrt{2} + 1 = x \in \mathbb{Q}$ is rational. Then $x - 1$ is also rational, since $\mathbb{Q}$ is closed under subtraction. But $x - 1 = \sqrt{2}$ is irrational, and this is a contradiction. Thus $\sqrt{2} + 1$ is irrational.

For (b), the statement is

$$\exists n \in \mathbb{Z} : \forall x, y \in \mathbb{R} : x^2 + y^2 \geq n.$$

The statement is true by taking $n = 0$, since $x^2 + y^2 \geq 0$ for all $x, y \in \mathbb{R}$.

For (c), we have $(-3) \cdot 13 = -39 \equiv 5 \pmod{11}$, so $[-3] \cdot [13] = [5]$ in $\mathbb{Z}/11\mathbb{Z}$.

Problem 9. For (a), the partitions are $\{\{-6, 0, 13\}\}$, $\{\{-6\}, \{0, 13\}\}$, $\{\{0\}, \{-6, 13\}\}$, $\{\{13\}, \{-6, 0\}\}$, $\{\{-6\}, \{0\}, \{13\}\}$: there are 5 partitions.

For (b), we see that the equivalence class of a must contain c and thus by transitivity it must contain d, and same with g, so $\{a, c, d, g\} \subseteq [a]$. Similarly, $\{b, f\} \subseteq [b]$. We need only to decide where e belongs; but since there are three equivalence classes, it must be alone in its equivalence class, since the equivalence classes form a partition of A. Thus the equivalence classes are

$$[a] = \{a, c, d, g\} \text{ and } [b] = \{b, f\} \text{ and } [e] = \{e\}.$$

For (c), we see that R is reflexive if and only if $3 \mid (a^3 - a)$ for all $a \in \mathbb{Z}$. This is true! Here is a proof. By the division algorithm, we have $a \equiv 0, 1, 2 \pmod{3}$. If $a \equiv 0 \pmod{3}$ then $a^3 \equiv 0^3 = 0 \equiv a \pmod{3}$; if $a \equiv 1 \pmod{3}$ then $a^3 \equiv 1^3 = 1 \equiv a \pmod{3}$; and if $a \equiv 2 \pmod{3}$ then $a^3 \equiv 2^3 = 8 \equiv 2 \pmod{3}$. In all three cases, we have $a^3 \equiv a \pmod{3}$, as claimed.

Problem 10. We show that $g \circ f$ is injective and surjective.

Let $x_1, x_2 \in A$ and suppose that $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then $g(f(x_1)) = g(f(x_2))$. Since g is injective, this implies that $f(x_1) = f(x_2)$. Since f is injective, this implies $x_1 = x_2$. Thus $g \circ f$ is injective.

Now let $z \in C$. Since g is surjective, there exists $y \in B$ such that $g(y) = z$. Since f is surjective, there exists $x \in A$ such that $f(x) = y$. Thus $(g \circ f)(x) = g(f(x)) = g(y) = z$, so $g \circ f$ is surjective.