MATH 241: ANALYSIS IN SEVERAL REAL VARIABLES I
FINAL EXAM

Problem 1. Let $a_n = 1/(tn+1)$. First, if $t = 0$, then the sequence is constant and $a_n = 1$ for all $n \in \mathbb{N}$: then, given any $\epsilon > 0$, we can take $N = 1$ and $|a_n - 1| = 0 < \epsilon$.

If $t \neq 0$, then we claim $a_n \to 0$. Let $\epsilon > 0$ and let $N > 1/(|t|\epsilon)$. Then if $n \geq N$ we have

$$\left| \frac{1}{tn+1} - 0 \right| < \frac{1}{|t|n} \leq \frac{1}{|t|N} < \epsilon.$$

Problem 2. Statement (a) is false, since the triangle inequality is invalid: we have $d(0,2) = 4 > d(0,1) + d(1,2) = 1 + 1 = 2$. Statement (b) is false: take for example $A_n = [1/(n+1), 1)$, then $\bigcup_{n=1}^{\infty} A_n = (0,1)$ (by the Archimedean property of $\mathbb{R}$, if you insist), which is open. Also, we can take $A_n = [-n, n]$ for then $\bigcup_{n=1}^{\infty} A_n = \mathbb{R}$ which is open. Statement (c) is true: there is a sequence (a_n) from A with $a_n \to x$, and by the topological characterization of convergence, every neighborhood of x contains all but finitely many elements of (a_n). Statement (d) is false, for example $x_n = 1/n^2$ and $y_n = n$ so that $x_n y_n = 1/n \to 0$. Statement (e) is true: if $x \in \mathbb{R}$, then there is a sequence $x_n \to x$ with $x_n \in \mathbb{Q}$ for all x since $\mathbb{Q}$ is dense in $\mathbb{R}$, so since f and g are continuous, we have $f(x) = \lim f(x_n) = \lim g(x_n) = g(x)$.

Problem 3. This was Exercise 6.2.7. Let $\epsilon > 0$. Since $f_n \to f$ uniformly, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon/3$ for all $n \geq N$ and all $x \in A$. Since f_N is uniformly continuous, there exists $\delta > 0$ such that $|f_N(x) - f_N(y)| < \epsilon/3$ for all $x, y \in A$ with $|x - y| < \delta$. Thus, for all $x, y \in A$ with $|x - y| < \delta$, we have

$$|f(x) - f(y)| < |f(x) - f_N(x)| + |f_N(x) + f_N(y)| + |f_N(y) - f(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

So f is uniformly continuous.

Problem 4. Statement (a) is false: a finite set is closed and bounded but not uncountable. Statement (b) is false: we need the sets to be nonempty and disjoint, and then it is a theorem. Statement (c) is true. We claim that $\#A \leq 1$, so A is finite so closed. Indeed, if $a, b \in A$ with $a < b$, then by the density of $\mathbb{Q}$ in $\mathbb{R}$ there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x < b$ and since A is connected we have $x \in A$, a contradiction since $A \subseteq \mathbb{Q}$. Statement (d) is false: take $f(x) = |x|$ if $x = 0$ and $f(0) = 1$. (We need f to be continuous.) Statement (e) is true: we have $\{x \in \mathbb{R} : f(x) > 0\} = f^{-1}(\mathbb{R}_{>0})$; since $\mathbb{R}_{>0}$ is open and f is continuous, $f^{-1}(\mathbb{R}_{>0})$ is open.

Problem 5. For (a), at $x = 0$, we claim that $f'(0) = 0$. We note that

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}.$$

Let $\epsilon > 0$. Let $\delta = \min(1, \epsilon/2)$. Then if $|x| < \delta$ then $|x + 1| \leq |x| + 1 < 2$, so

$$\left| \frac{f(x)}{x} \right| = |x^2 + x| = |x||x + 1| < 2|x| < \epsilon.$$

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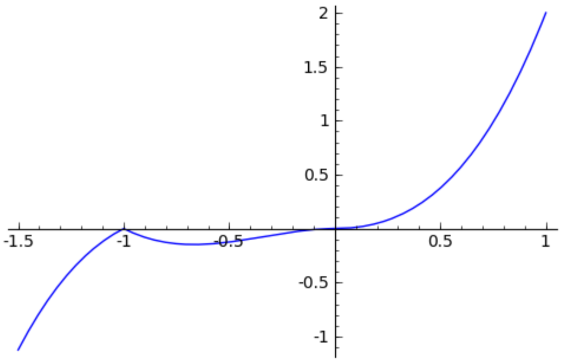

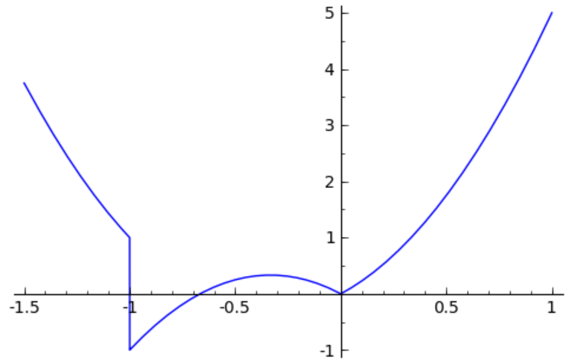

For (b), we compute that

$$f(x) = x|x(x+1)| = \begin{cases} -(x^3+x^2), & \text{if } -1 < x < 0; \\ x^3+x^2, & \text{if } x > 0. \end{cases}$$

Thus

$$\frac{f'(x)}{x} = \begin{cases} -(3x+2), & \text{if } -1 < x < 0; \\ 3x+2, & \text{if } x > 0. \end{cases}$$

We claim that $f'(x)$ is not differentiable at $x = 0$. Let $x_n = 1/n$ and $y_n = -1/n$ for $n \in \mathbb{N}_{\geq 1}$. Then $\lim f'(x_n)/x_n = \lim -(3x_n+2) = -2$ by the algebraic limit theorem, but $\lim f'(y_n)/y_n = \lim 3(-1/n)+2 = 2$. By the sequential criterion, $f''(0) = \lim f'(x)/x$ does not exist.

Problem 6. Note that f is not a power series. Let $f_n(x) = x^n \cos(nx)$. For (a), we have $|f_n(x)| = |x^n \cos(nx)| \leq (1/2)^n = M_n$ and $\sum_{n=1}^{\infty} 1/2^n$ is a convergent geometric series, so by the Weierstrass M-test, the series converges uniformly so by our theorem the function f is continuous on $[-1/2, 1/2]$.

For (b), we have $f_n'(x) = nx^{n-1}\cos(nx) - nx^n \sin(nx) = nx^{n-1}(\cos(nx) + x\sin(nx))$. We have

$$|f_n'(x)| < n(1/2)^{n-1}(1+1/2) = 3n/2^n = M_n.$$

The series $\sum_{n=1}^{\infty} 3n/2^n$ converges by the ratio test, since

$$\lim \frac{3(n+1)}{2^{n+1}} \cdot \frac{2^n}{3n} = 1/2 < 1.$$

Or, this series is 3/2 times the value of the power series $\sum_{n=1}^{\infty} nx^{n-1} = (1/(1-x))' = 1/(1-x)^2$ at $x = 1/2$; term-by-term differentiation is valid by our classwork (for what it's worth, we obtain an exact value for the series in this way). By the Weierstrass M-test again, the series converges.

Problem 7. Let $g(x) = f(x) - f(x-1)$. Then g is continuous on $[1,2]$ and $g(1) = f(1) - f(0) < 0$ and $g(2) = f(2) - f(1) > 0$. By the intermediate value theorem, there exists $x \in (1,2)$ such that $g(x) = f(x) - f(x-1) = 0$, so $f(x) = f(y)$ where $y = x - 1$ so $x - y = 1$.

Problem 8. For part (a), we have $(\sqrt{x} - \sqrt{y})^2 = x - 2\sqrt{xy} + y \geq 0$ so $(x+y)/2 \geq \sqrt{xy}$, with equality if and only if $x = y$.

For part (b), first we have $a_n \leq b_n$ by part (a). Thus $a_{n+1} = \sqrt{a_n b_n} \geq \sqrt{a_n^2} = a_n$ for all n, so (a_n) is increasing. Similarly, $b_{n+1} = (a_n + b_n)/2 \leq (b_n + b_n)/2 = b_n$ for all n so (b_n) is decreasing. The sequence (a_n) is bounded above by b_1 since $a_1 \leq a_n \leq b_n \leq b_1$ and similarly (b_n) is bounded below by a_1, so by the monotone convergence theorem, the sequences (a_n) and (b_n) are convergent.

For part (c), if $\lim a_n = a$ and $\lim b_n = b$, we have $b = \lim b_{n+1} = \lim(a_n + b_n)/2 = (a+b)/2$ by the algebraic limit theorem, so $a = b$.

Problem 9. Suppose $|f'(c)| \leq M$ for all $c \in \mathbb{R}$. For all $x, y \in \mathbb{R}$ with $x \neq y$, by the mean value theorem, there exists $c \in \mathbb{R}$ such that $(f(x) - f(y))/(x-y) = f'(c)$, so $|f(x) - f(y)|/|x-y| = |f'(c)| \leq M$.

Let $\epsilon > 0$ and let $\delta = \epsilon/M$. Then, if $|x-y| < \delta$ (and $x \neq y$) we have

$$|f(x) - f(y)| = \frac{|f(x) - f(y)|}{|x-y|}|x-y| < M\frac{\epsilon}{M} = \epsilon.$$