MATH 241: ANALYSIS IN SEVERAL REAL VARIABLES I
FINAL EXAM

Problem 1. Part (a) is false: take $f(x) = \cos(1/x)$ and $x_n = 1/(\pi n)$, so that $x_n \to 0$ but $f(x_n) = (-1)^n$. Part (b) is false by Darboux's theorem: a differentiable function has a derivative which satisfies the intermediate value property. Statement (c) is false: Take $A = \mathbb{R} \setminus \{\sqrt{2}\}$, which is open since the complement $\{\sqrt{2}\}$ is closed, and $A \supset \mathbb{Q}$ since $\sqrt{2}$ is irrational. Statement (d) is false: take $[0,1] \cup [2,3]$. Statement (e) is also false: the function $f(x) = x^2/(1+x^2)$ is continuous and bounded but has no maximum value (it approaches but never achieves 1 as $x \to \infty$).

Problem 2. We claim $n/(2n+1) \to 1/2$. Let $\epsilon > 0$. Let $N > 1/(4\epsilon)$. Then for $n \geq N$ we have

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| = \frac{1}{2(2n+1)} < \frac{1}{4n} \leq \frac{1}{4N} < \epsilon$$

as claimed.

Problem 3. Let $s = \sup A$ and $t = \sup B$. Then st is an upper bound for AB, since if $ab \in AB$ then $a \in A$ and $b \in B$ so $a \leq s$ and $b \leq t$ so $ab \leq st$. By the axiom of completeness, AB has a least upper bound, say $u = \sup AB$. Then $u \leq st$ by the above. Since u is an upper bound, we have for all $a \in A$ and $b \in B$ that $ab \leq u$. But then $a \leq u/b$ for all $a \in A$ (and any $b \in B$) so $s \leq u/b$ since $s = \sup A$, and hence $b \leq u/s$. But then $t \leq u/s$, so $st \leq u$. Thus $u = st$.

Problem 4. If the sequence was convergent with $a_n \to a > 0$, then

$$a = \lim(a_n) = \lim(a_{n+1}) = \lim\left(\frac{a_n}{2} + \frac{5}{a_n}\right) = \frac{a}{2} + \frac{5}{a}$$

by the algebraic limit theorem, so $a^2 = 10$. Since $a_n > 0$, by the order limit theorem, we must have $a = \sqrt{10}$.

To show that the sequence converges we apply the monotone convergence theorem: we show it is (eventually) decreasing and bounded below. We claim that $a_n > \sqrt{10}$ for $n \geq 2$: we have

$$a_{n+1} = \frac{a_n}{2} + \frac{5}{a_n} = \frac{a_n^2 + 10}{2a_n} > \sqrt{10}$$

if and only if $(a_n - \sqrt{10})^2 > 0$, which is true. To show it is decreasing we note that $a_{n+1} = \dfrac{a_n}{2} + \dfrac{5}{a_n} \leq a_n$ if and only if $a_n^2 \geq 10$, which holds by the previous statement.

Problem 5. Statement (a) is true: the set $\{x + 1/n : x = 0,1,2,3 \text{ and } n \in \mathbb{N}\}$ has the limit points $0,1,2,3$. Statement (b) is false: finite sets are neither countable nor uncountable. Statement (c) is false: take $f_n(x) = g_n(x) = x$ on $\mathbb{R}$: then trivially each is uniformly convergent, but we showed in class that x^2 is not. Statement (d) is true: if A is compact then A is closed, and since f is continuous $f^{-1}(A)$ is closed. Statement (e) is also true: since A is compact and f is continuous then $f(A)$ is compact, hence closed.

Problem 6. Let $\sum_{n=1}^{\infty} a_n = A$. Then by the Cauchy criterion there exists $N \in \mathrm{N}$ such that for all $n > m \geq N$ we have $|a_n + a_{n-1} + \cdots + a_{m+1}| = a_n + \cdots + a_{m+1} < \epsilon/2$. Let $m = N$. Then by the order limit theorem (on the partial sums) we obtain

$$a_{N+1} + a_{N+2} + \cdots = \sum_{k=N+1}^{\infty} a_k \leq \frac{\epsilon}{2} < \epsilon.$$

Let $b_n = a_{N+n}$, so e.g. $b_1 = a_{N+1}$. Then $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_{n+N} = \sum_{k=N+1}^{\infty} a_k < \epsilon$, as desired.

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Problem 7. First, part (a). By calculus, f is differentiable for all $x \neq 0$. We claim that $f'(0) = 1$, which is to say we prove that
$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x) - 1}{x} = 1.$$
Let $\epsilon > 0$ and let $\delta = \epsilon$. Let $|x| < \delta$. If $x > 0$ then $\left|\frac{f(x)-1}{x} - 1\right| = 0$ identically. If $x < 0$ we have
$$\left|\frac{f(x)-1}{x} - 1\right| = \left|\frac{1}{1-x} - 1\right| = \left|\frac{x}{1-x}\right| < |x| < \delta = \epsilon.$$
Therefore indeed $f'(0) = 1$. For part (b), we have that
$$f'(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ \dfrac{1}{(1-x)^2}, & \text{if } x < 0. \end{cases}$$
So indeed $f'(x)$ is continuous, since $\lim_{x\to 0} 1/(1-x)^2 = 1$. (Note that f' itself is not differentiable, though.)

Problem 8. For any $x \in [0, 1)$, we proved in class that $x^n \to 0$, so $f_n(x) \to 0$ by the algebraic limit theorem. For $x = 1$ we have $f_n(1) = 1/2$ identically, so $f_n(x)$ converges pointwise to
$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1; \\ 1/2, & \text{if } x = 1. \end{cases}$$
The convergence is not uniform: if it were, then since each $f_n(x)$ is continuous so too would be $f(x)$.

Problem 9. Since $\sum_{n=1}^{\infty} a_n$ is convergent, we have $|a_n| \to 0$. Therefore, there exists $N \in \mathrm{N}$ such that $|a_n| < 1$ for $n \geq N$. Thus $a_n^2 < |a_n|$ for $n \geq N$. Therefore
$$\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{N} a_n^2 + \sum_{n=N+1}^{\infty} a_n^2 < \sum_{n=1}^{N} a_n^2 + \sum_{n=N+1}^{\infty} |a_n|.$$
The latter sum converges since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and therefore by the comparison test the series $\sum_{n=1}^{\infty} a_n^2$ converges.

Problem 10. Consider $g(x) = f(x) - x^2$. Then $g : [-1, 2] \to \mathbb{R}$ is a continuous function and $g(-1) = f(-1) - (-1)^2 = -1$ and $g(2) = f(2) - 2^2 = 5 - 4 = 1$. By the intermediate value theorem, there exists $x \in (-1, 2)$ such that $g(x) = f(x) - x^2 = 0$, so $f(x) = x^2$.

Problem 11. Suppose that $|f'(c)| \leq M$ for all $c \in (a, b)$. Now let $\epsilon > 0$, and let $\delta = \epsilon/M$. Let $x, y \in [a, b]$ with $x < y$ and $|x - y| < \delta$. Then by the mean value theorem, there exists $c \in (x, y)$ such that $\frac{f(x) - f(y)}{x - y} = f'(c)$. So $\left|\frac{f(x) - f(y)}{x - y}\right| = |f'(c)| \leq M$ so $|f(x) - f(y)| \leq M|x - y| < M\delta = M(\epsilon/M) = \epsilon$.

Problem 12. By the extreme value theorem, f achieves its minimum and maximum. We claim that the minimum is achieved at $x = a$ and the maximum at $x = b$. We prove the latter, the former follows by instead considering $-f$. Suppose the maximum L is achieved at $c \in [a, b]$. Since $f(a) < f(b)$, we have $c \in (a, b]$. Note that $L \geq f(b)$. If $c = b$ then we are done, so suppose that $c \in (a, b)$. Then by the intermediate value theorem, f restricted to the interval $[a, c]$ achieves all values between $f(a)$ and $L \geq f(b)$, so in particular there exists $d \in [a, c]$ such that $f(d) = L = f(b)$. But this contradicts the fact that f is one-to-one.