Integer-Point Enumeration in Polyhedra

Based on "Computing the Continuous Discretely"
by Matthias Beck and Sinai Robins

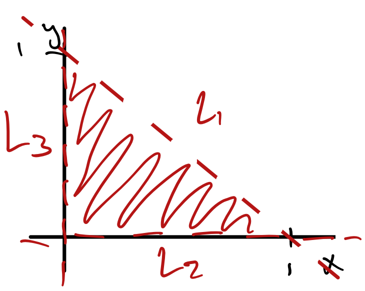
What is a Polyhedron?

Polyhedron: A set, bounded or not, that can be described by the intersection of finitely many half-spaces and hyperplanes

Convex Polytope: A convex hull of finitely many points in $\mathbb{R}^d$.

- Equivalent to a bounded polyhedron

Ex. Polyhedron in $\mathbb{R}^2$



- Defined by hyperplanes:

L_1: $\{(x,y) \in \mathbb{R}^2 \mid y = -x+1\}$

L_2: $\{(x,0) \in \mathbb{R}^2\}$

L_3: $\{(0,y) \in \mathbb{R}^2\}$

$P = \{(x,y) \in \mathbb{R}^2 \mid y < -x+1,\ y > 0,\ x > 0\}$

If we include the lines L_1, L_2, L_3, then P becomes the convex polytope defined by

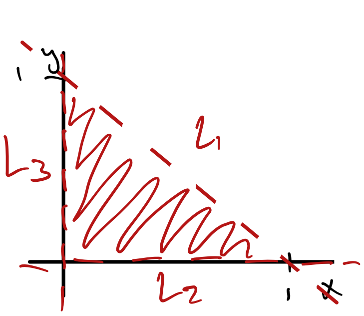
$$P = \text{Conv}\{(0,0),(0,1),(1,0)\}$$

What is the dimension of a Polytope?

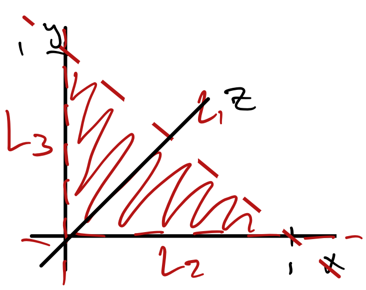
$$\dim P = \dim(\text{Span } P = \{x + \lambda(y-x) \mid x,y \in P, \lambda \in \mathbb{R}\}$$

- The smallest space our polytope can comfortably reside in

Ex.



dim = ?



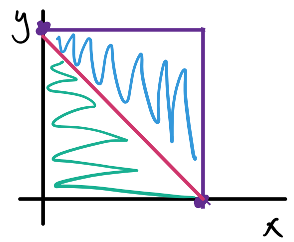
dim = ?

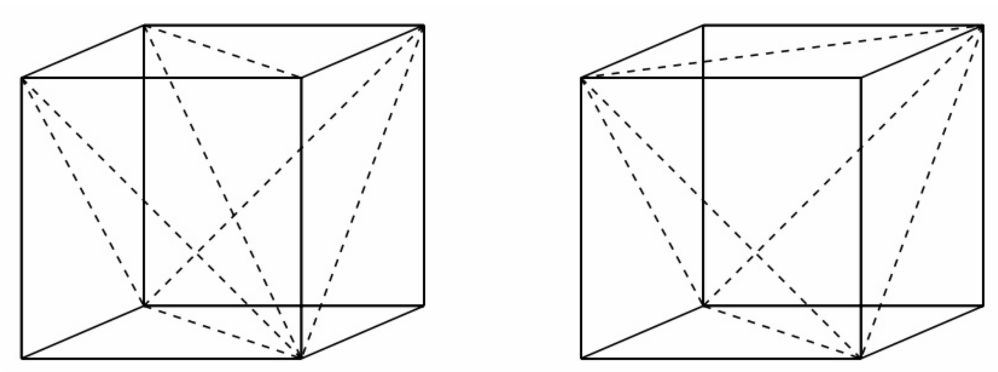
Fact: a d-dimensional polytope has at least d+1 vertices.

Simplex: a d-dimensional polytope with exactly d+1 vertices

Triangulating Over a Polytope:

Triangulation: "Splitting up" a polytope into simplices of the same dimension





• Beck, Robins, p60

Fact: Triangulations are not necessarily unique

Thm: Every convex polytope can be triangulated using no new vertices

Why do we care about Triangulation?

It makes all of our proofs easier!

Integer-Point Enumeration

$\mathbb{Z}^d = \{(a_1, \dots, a_d) \mid a_i \in \mathbb{Z}\}$ is the "d-integer lattice"

- A grid w/ a point whenever all coordinates are integers

Lattice-Point Enumeration (L_P):

$$L_P := \#(P \cap \mathbb{Z}^d)$$

- The number of integer points in P.
- "discrete volume"

Dilation

We can scale our polytope P by t

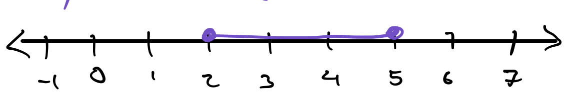
- tP is called the "t^{th} dilate" of P

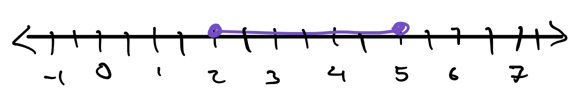
$$L_P(t) = \#(tP \cap \mathbb{Z}^d) = \#(P \cap \tfrac{1}{t}\mathbb{Z}^d)$$

Examples

$d = 1$: $P = [a, b]$, $a, b \in \mathbb{Z}$

$P = [2, 5]$

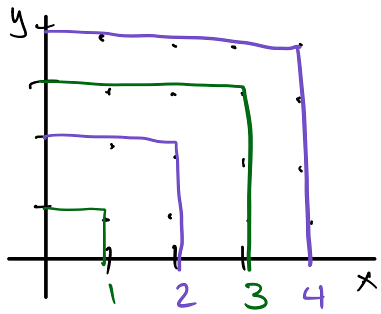




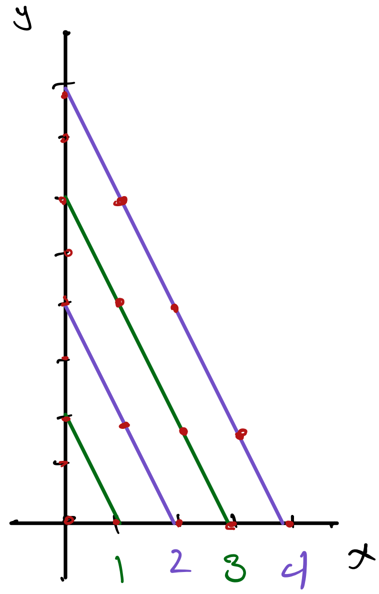
$L_{[2,4]}(1) = 4$

$L_{[2,4]}(2) = 7 \neq 2(4)$

$d = 2$:



$L_P(t) = (t+1)^2$



$L_P(t) = (t+1)^2$

$\Rightarrow$ $L_P(t)$ is not unique for a given polytope

Why study integer-point enumeration?

There is a deep connection between discrete and continuous volume

Discrete $\to$ Continuous

For a polytope $P \subset \mathbb{R}^d$:

- Draw d-dimensional boxes between the points of your integer lattice.
- # of Boxes in $P \approx h_P(1)$
- Volume of each box $= 1^d = 1$
- Very rough volume approximation of $P = 1 \cdot h_P(1)$

What if we dilate P by scaling our lattice down?

- # of boxes in $P \approx h_P(t)$
- Volume of each box $= \frac{1}{t^d}$
- $\mathrm{Vol}(P) \approx \frac{1}{t^d}(h_P(t))$

$$\mathrm{Vol}(P) = \lim_{t\to\infty} \frac{1}{t^d}(h_P(t))$$

- What does this tell us about $O(h_P(t))$?

Frobenius's Coin Problem

Given n coins of different values, what's the largest number we can't make using these coins?

Coins: $a_1, \ldots, a_d \in \mathbb{Z}_{>0}$

$$P_A(n) = \#\{(m_1, \ldots, m_d) \in \mathbb{Z}^d \mid m_1, \ldots, m_d \geq 0,\ a_1 m_1 + a_2 m_2 + \cdots + a_d m_d = n\}$$

- Counts how many ways we can make n w/ our coins

The Ehrhart Series

$$\mathrm{Ehr}_P(z) := 1 + \sum_{t \geq 1} L_P(t) z^t$$

** **Ehrhart's Thm:**

If P is an integral convex d-polytope, then $L_P(t)$ is a polynomial of degree d.

Proof Outline:

Lemma:

If $\displaystyle\sum_{t \geq 0} f(t) z^t = \frac{g(z)}{(1-z)^{d+1}}$

then f is a polynomial of degree d

$$\iff$$

g is a polynomial of degree $\leq d$, $g(1) \neq 0$

Coning over a polytope

- Integer-point transform: $\sigma_S(\mathbf{z}) = \sigma_S(z_1, z_2, \dots, z_d) = \sum_{\mathbf{m} \in S \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{m}}$

Calculating $\mathrm{Ehr}_{\mathcal{P}}(z)$

If $\mathrm{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{t \geq 1} f(t) z^t = \frac{g(z)}{(1-z)^{d+1}}$

then our Ehrhart Series is uniquely determined by $g(z)$, a polynomial of degree $\leq d$.

For Integral Convex d-Polytopes:

Let $g(z) = h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_1^* z + h_0^*$

$$\Rightarrow h_0^* = 1$$

$$\Rightarrow h_1^* = L_{\mathcal{P}}(1) - d - 1$$

Similarly, there are formulas for $h_2^*, h_3^*, \ldots$ based on $L_{\mathcal{P}}(2), L_{\mathcal{P}}(3), \ldots$

So we can calculate $g(z)$, and therefore $\mathrm{Ehr}_{\mathcal{P}}(z)$ by looking at the first d dilates of $\mathcal{P}$.

We can actually do better!

Theorem:

If P is an integral convex d-polytope with

$$\mathrm{Ehr}_P(z) = \frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_1^* z + 1}{(1-z)^{d+1}}$$

Then $h_d^* = h_{d-1}^* = \cdots = h_{k+1}^* = 0$

$$\iff$$

$(d-k+1)P$ is the smallest integer dilate of P that contains an interior lattice point.

Proof Outline

Ehrhart-Macdonald reciprocity:

Suppose P is a convex rational polytope

$\Rightarrow L_P(-t) = (-1)^{\dim(P)} L_{P^\circ}(t)$

where P° is the interior of P.

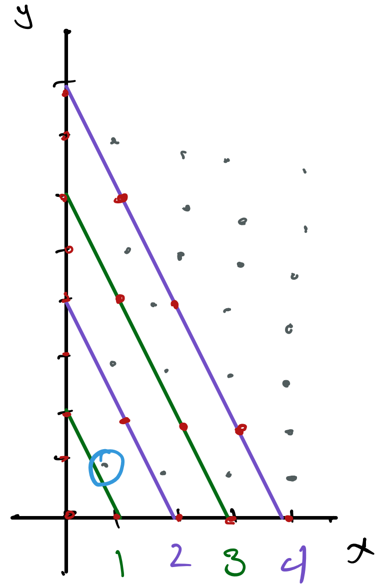
Big Takeaway:

To calculate the Ehrhart Series for a Polytope $\mathcal{P}$, we only need to look at the dilates of $\mathcal{P}$ until we find an integer lattice point in the interior (or at worst d of them)

Remember:

Since $\text{Ehr}_{\mathcal{P}}(z)$ is encoded by $L_{\mathcal{P}}(t)$, we can find $L_{\mathcal{P}}(t)$ for any dilate t by looking at the coefficient of z^t in the Ehrhart Series

Ex:



$$Ehr_P(z) = \frac{g(z)}{(1-z)^3} = \frac{h_2^* z^2 + h_1^* z + 1}{(1-z)^3}$$

$$h_1 = h_P(1) - d - 1 = 4 - 2 - 1 = 1$$

$$d - k + 1 = 2$$

$$2 - k + 1 = 2$$

$$\boxed{k = 1}$$

$$h_p(2) = \binom{d+2}{d} + h_1^* \binom{d+1}{d} + h_2^* \binom{d}{d}$$

$$9 = \binom{4}{2} + 1\binom{3}{2} + h_2^*(1)$$

$$9 = 6 + 3 + h_2^*$$

$$\Rightarrow h_2^* = 0$$

What about Rational Polytopes?

Quasipolynomials: Periodic functions that alternate between polynomials (constituents)

Degree of a quasipolynomial $Q(t)$ is the highest degree among its constituents

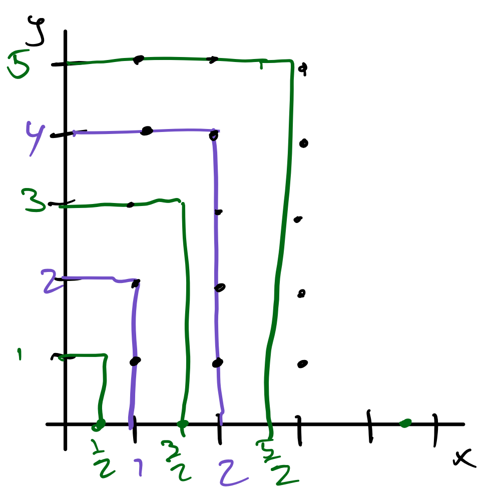
Ehrhart's Theorem for Rational Polytopes

If P is a <u>rational</u> convex d-polytope

Then $L_P(t)$ is a quasipolynomial in t of degree d.

The period of Q divides the lcm of the denominators of the coordinates of the vertices of P

Ex:



t	$L_P(t)$
1	2
2	6
3	8
4	12
5	15

$$L_P(t) = \begin{cases} (t+1)\left(\frac{t+1}{2}\right) : t \text{ odd} \\ (t+1)\left(\frac{t+2}{2}\right) : t \text{ even} \end{cases}$$