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Proper actions on imprimitivity bimodules and decompositions of Morita equivalences ☆

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Abstract

We consider a class of proper actions of locally compact groups on imprimitivity bimodules over C^*-algebras which behave like the proper actions on C^*-algebras introduced by Rieffel in 1988. We prove that every such action gives rise to a Morita equivalence between a crossed product and a generalized fixed-point algebra, and in doing so make several innovations which improve the applicability of Rieffel's theory. We then show how our construction can be used to obtain canonical tensor-product decompositions of important Morita equivalences. Our results show, for example, that the different proofs of the symmetric imprimitivity theorem for actions on graph algebras yield isomorphic equivalences, and this gives new information about the amenability of actions on graph algebras.

1. Introduction

A Morita equivalence between two C^*-algebras A and B is implemented by an imprimitivity bimodule ${}_AX_B$, which carries the structure necessary to induce Hilbert-space representations from B to A and back again. There are often several ways of

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constructing these bimodules, and, unsurprisingly, some ways are better for some things, and others for others. One therefore wants to be able to switch between different bimodules implementing equivalences between the same algebras.

To illustrate the kind of problems which arise, we consider a situation which underlies many important equivalences. Suppose we have commuting free and proper actions of locally compact groups K and H on the left and right of a locally compact Hausdorff space P. The orbit spaces P/H and $K\backslash P$ are again locally compact Hausdorff spaces, and carry actions of, respectively, K and H; the symmetric imprimitivity theorem of Green and Rieffel states that the crossed products $C_0(P/H) \rtimes K$ and $C_0(K\backslash P) \rtimes H$ are Morita equivalent. In the original proof of [16], a suitable imprimitivity bimodule Z was constructed by completing the space $C_c(P)$ of continuous functions of compact support. It was later shown in [2] that one could appeal to a previous construction of Green which gives $C_0(P/H)$–$(C_0(P) \rtimes H)$ and $(C_0(P) \rtimes K)$–$C_0(K\backslash P)$ bimodules X and Y, form crossed product bimodules $X \rtimes K$ and $Y \rtimes H$, and take the internal tensor product $(X \rtimes K) \otimes_{C_0(P) \rtimes (K \times H)} (Y \rtimes H)$ as the desired $(C_0(P/H) \rtimes K)$–$(C_0(K\backslash P) \rtimes H)$ imprimitivity bimodule. This latter construction has advantages: for example, it saves burrowing into the detailed construction of bimodules, allows us to analyze the effect of extra structure in stages, and makes it easier to prove analogues for reduced crossed products. On the other hand, we have a concrete bimodule Z, which is much more convenient for direct calculations. To make the best of both worlds, we need to prove that

$$Z \cong (X \rtimes K) \otimes_{C_0(P) \rtimes (K \times H)} (Y \rtimes H) \tag{1.1}$$

as $(C_0(P/H) \rtimes K)$–$(C_0(K\backslash P) \rtimes H)$ imprimitivity bimodules.

We ran into problems like these in [7], where we found an isomorphism implementing (1.1) using ad hoc methods; to verify that it worked, we had to do awful calculations involving quintuple integrals. One goal of the present project was to find a more systematic way of identifying and verifying such isomorphisms: our Theorem 3.1 tells us not just that there is an isomorphism, but also how to write it down.

To make our approach as systematic as possible, we have worked within the general framework of proper actions of groups on C^*-algebras, as developed by Rieffel in [17], and we have, we hope, made significant improvements to that theory. In particular, we have extended Rieffel's main Morita equivalence in [17, Corollary 1.7] to cover proper actions on imprimitivity bimodules. This extension turned out to be anything but routine, and we are optimistic that some of the technical tools we have developed will help in constructing Morita equivalences for more general integrable actions, where substantial technical problems arise (see [18, Section 6]). Because Rieffel's framework involves reduced crossed products rather than full ones, our main results are about reduced crossed products. We intend to apply our techniques to full crossed products elsewhere. We emphasize that our notion of proper action is related to Rieffel's original version in [17] rather than the integrable actions studied recently in [5,10,18].

We begin in Section 2 by discussing proper actions on imprimitivity bimodules. We start with a Morita equivalence (X, G, γ) between two C^*-dynamical systems (A, G, α) and (B, G, β). The action γ is *proper* if there is a γ-invariant pre-imprimitivity bimodule ${}_{A_0}(X_0)_{B_0} \subset X$ with properties like those of the dense subalgebra used by Rieffel. There are several ways in which the technical hypotheses could be phrased; we have chosen one which reduces to that of [17] when ${}_AX_B = {}_BB_B$, bears a striking formal resemblance to it, and yields the desired Morita equivalence $\overline{X_0}$ between $A \rtimes_{\alpha,r} G$ and a generalized fixed-point algebra B^β when the action is also saturated (Theorem 2.16). The proof of Theorem 2.16, though, is quite different from its analogue in [17]. For ${}_AX_B = {}_BB_B$, Rieffel proved that the $(B \rtimes_{\beta,r} G)$-valued inner product has the required properties, that B^β acts as adjointable operators on the resulting left Hilbert module ${}_{B \rtimes_r G}Z$, and then that the map $B^\beta \to \mathcal{L}({}_{B \rtimes_r G}Z)$ is isometric [17, p. 151]. We were not able to extend this last part, so we had to substantially reshape the whole argument, starting with the right inner product rather than the left. In retrospect, this is probably a good thing. The process we have gone through is similar to the program discussed by Rieffel in his later paper [18, Section 6], and since we have been able to sidestep some of the general problems he raises in our setting, our arguments may be useful in the more general context. Indeed, we have already used some of these ideas to find new insight on how the symmetric imprimitivity theorem relates to reduced crossed products (see [6]).

In Section 3, we prove our general decomposition theorem. The key idea is that one obtains a decomposition like (1.1) whenever one has a Morita equivalence for the linking algebra $L(X)$ of another Morita equivalence; the key Lemma 3.2 is a one-sided version of a result from [3]. The main work in Section 3 is to show that if the action γ of G on X is proper and saturated, then so are the associated actions on B and $X \oplus B$; we then apply Lemma 3.2 to a bimodule over $L(X \rtimes G)$ arising from an application of Theorem 2.16 to $X \oplus B$. The result is a tensor-product decomposition for the bimodule $\overline{X_0}$ of Theorem 2.16, which in the situation of the symmetric imprimitivity theorem turns out to be the desired isomorphism (1.1).

In the last section, we apply Theorem 3.1 to crossed products of graph algebras. This gives new information about the symmetric imprimitivity theorem for graph algebras, and allows us to settle a question left open in [11].

1.1. Background on integration

Let G be a locally compact group, A a C^*-algebra and $f : G \to A$ a continuous function such that $\int_G \|f(s)\|\, ds < \infty$. Minor modifications to the construction of [15, Lemma C.3], for example, show that f has an integral $\int_G f(s)\, ds$, and that bounded linear maps and multipliers pull through the integral. By viewing a Hilbert module X as a corner in its linking algebra $L(X)$, we can extend this integral and its properties to functions $f : G \to X$ such that $\int_G \|f(s)\|\, ds < \infty$.

2. Proper actions on imprimitivity bimodules

Throughout this section, (X, G, γ) will be a Morita equivalence between two dynamical systems (A, G, α) and (B, G, β); since there is only one locally compact group G involved, we will drop it from our notation.

Definition 2.1. The action γ of G on ${}_AX_B$ is *proper* if there are an invariant subspace X_0 of X and invariant $*$-subalgebras A_0 of A and B_0 of B, such that ${}_{A_0}(X_0)_{B_0}$ is a pre-imprimitivity bimodule with completion ${}_AX_B$, and such that

(1) for every $x, y \in X_0$, the functions $s \mapsto \Delta(s)^{-1/2}\ {}_A\langle x, \gamma_s(y)\rangle$ and $s \mapsto\ {}_A\langle x, \gamma_s(y)\rangle$ are in $L^1(G, A)$;
(2) for every $b \in B_0$ and $x \in X_0$, the functions $s \mapsto \gamma_s(x) \cdot b$ and $s \mapsto \Delta(s)^{-1/2}\gamma_s(x) \cdot b$ are in $L^1(G, X)$;
(3) for every $x, y \in X_0$, there is a multiplier $\langle x, y\rangle_D$ in $M(B_0)^\beta$ such that $z \cdot \langle x, y\rangle_D \in X_0$ for all $z \in X_0$, and

$$\int_G b\beta_s(\langle x, y\rangle_B)\, ds = b\langle x, y\rangle_D \quad \text{for all } b \in B_0. \tag{2.1}$$

That the integral in (2.1) exists follows from

$$\|b\beta_s(\langle x, y\rangle_B)\| = \|\langle \gamma_s(y), \gamma_s(x) \cdot b^*\rangle_B\| \leqslant \|\gamma_s(x) \cdot b^*\|\, \|y\|$$

and item (2) of the definition.

Remark 2.2. (1) There are some subtleties to this definition. First, asserting that ${}_{A_0}(X_0)_{B_0}$ is a pre-imprimitivity bimodule is an efficient way of saying many things; for example, it implies that $x \cdot b \in X_0$ whenever $x \in X_0$ and $b \in B_0$, and ${}_A\langle x, y\rangle \in A_0$ whenever $x, y \in X_0$. Second, saying that X is the completion of X_0 is meant to include that A_0 is dense in A and B_0 is dense in B. Third, the D adorning the inner product does not yet exist: it will be defined in Proposition 2.3 below, and proved there that $\langle \cdot, \cdot \rangle_D$ is a D-valued inner product.

(2) Note that the action β of G on ${}_BB_B$ is proper with respect to ${}_{B_0}(B_0)_{B_0}$ if and only if the action β on B is proper with respect to B_0 in the sense of [17, Definition 1.2].

(3) Definition 2.1 is not symmetric: asserting that γ is proper is not the same as asserting that the action $\tilde{\gamma}$ on the dual equivalence ${}_B\tilde{X}_A$ is proper.

We will prove that if the Morita equivalence ${}_{(A,\alpha)}(X, \gamma)_{(B,\beta)}$ is proper with respect to ${}_{A_0}(X_0)_{B_0}$, then X_0 completes to a Morita equivalence between an ideal E of $A \rtimes_{\alpha,r} G$ and a generalized fixed-point algebra $D \subset M(B_0)^\beta$ of B.

Proposition 2.3. *Let D be the closure of $D_0 := \operatorname{sp}\{\langle x, y\rangle_D : x, y \in X_0\}$ in $M(B)$, where $\langle \cdot, \cdot \rangle_D$ is defined by Definition* 2.1(3). *Then D is a C^*-algebra, and $\langle \cdot, \cdot \rangle_D$ is a D-valued inner product on X_0.*

Proof. Let $x, y \in X_0$ and $d \in D_0$. Then $d \in M(B_0)^\beta$, and so for every $b \in B_0$, we have

$$\begin{aligned} b\langle x, y\rangle_D\ d &= \left(\int_G b\beta_t(\langle x, y\rangle_B)\, dt\right) d = \int_G b\beta_t(\langle x, y\rangle_B\ d)\, dt \\ &= \int_G b\beta_t(\langle x, y \cdot d\rangle_B)\, dt = b\langle x, y \cdot d\rangle_D, \end{aligned}$$

which implies $\langle x, y\rangle_D\ d = \langle x, y \cdot d\rangle_D$; since we know from Definition 2.1(3) that $y \cdot d \in X_0$, this implies that D_0 is an algebra. Similarly, we have

$$\begin{aligned} b\langle y, x\rangle_D &= \int_G b\beta_s(\langle y, x\rangle_B)\, ds = \int_G b\beta_s(\langle x, y\rangle_B)^*\, ds \\ &= \left(\int_G \beta_s(\langle x, y\rangle_B) b^*\, ds\right)^* = (\langle x, y\rangle_D\ b^*)^* = b\langle x, y\rangle_D^*, \end{aligned}$$

which implies that $\langle y, x\rangle_D = (\langle x, y\rangle_D)^*$. This proves both that D_0 is a $*$-algebra, so its closure D is a C^*-algebra, and that $\langle \cdot, \cdot \rangle_D$ has the algebraic properties of an inner product.

To show positivity of $\langle \cdot, \cdot \rangle_D$, let π be a faithful nondegenerate representation of B, and note that

$$\begin{aligned} \langle x, x\rangle_D \geqslant 0 &\iff \langle x, x\rangle_D \geqslant 0 \quad \text{in } M(B) \\ &\iff (\bar{\pi}(\langle x, x\rangle_D)h \mid h) \geqslant 0 \quad \text{for all } h \in \mathscr{H}_\pi. \end{aligned}$$

Since B_0 is dense in B and π is nondegenerate, it is enough to show this when $h = \pi(b)k$ for $b \in B_0$ and $k \in \mathscr{H}_\pi$. Well,

$$\begin{aligned} (\bar{\pi}(\langle x, x\rangle_D)\pi(b)k \mid \pi(b)k) &= (\pi(b^*\langle x, x\rangle_D\ b)k \mid k) \\ &= \left(\pi\left(\int_G b^*\beta_s(\langle x, x\rangle_B) b\, ds\right)k \;\middle|\; k\right) \\ &= \int_G (\pi(b^*\beta_s(\langle x, x\rangle_B) b)k \mid k)\, ds, \end{aligned}$$

which is positive because each $b^*\beta_s(\langle x, x\rangle_B)b$ is positive. Because $b^*\beta_s(\langle x, x\rangle_B)b$ is continuous in s, this calculation also shows that

$$\langle x, x\rangle_D = 0 \Rightarrow \beta_s(\langle x, x\rangle_B) = 0 \quad \text{for all } s \Rightarrow x = 0,$$

so $\langle \cdot, \cdot \rangle_D$ is definite. □

Remark 2.4. To prove that an element d of a C^*-algebra D is positive, it is usually enough to take a faithful representation μ of D and prove that $\mu(d) \geqslant 0$ as an operator on $\mathscr{H}_\mu$. For the above argument, however, it is essential that the representation μ of D_0 is the restriction of (an extension of) a representation π of B. In general, not every representation μ of D_0 arises this way, and choosing $\mu = \bar{\pi}|_{D_0}$ means that we are proving $d \geqslant 0$ in the C^*-algebra obtained by completing D_0 in the norm of the C^*-algebra $M(B)$. This observation is crucial in [6].

Lemma 2.5. *There are homomorphisms* μ *of* A *into* $\mathscr{L}(\overline{(X_0)_D})$ *and* U *of* G *into* $U\mathscr{L}(\overline{(X_0)_D})$ *such that* $\mu(a)x = a \cdot x$, $\quad U_s x = \Delta(s)^{1/2}\gamma_s(x)$ $\quad$ *and* $\mu(\alpha_s(a)) = U_s \mu(a) U_s^*$ *for* $a \in A_0$, $x \in X_0$.

Remark 2.6. The homomorphism $\gamma : G \to \mathscr{L}(X_B)$ is not unitary: it changes the inner product by β_s. So it is important here that we are talking about the D-valued inner product on X_0, and we have had to introduce the modular function to ensure that U_s preserves this inner product. We are not asserting that (μ, U) is a covariant representation in the usual sense: neither nondegeneracy of μ nor continuity of U seems obvious. We shall return to this point in Lemma 2.17 below. Meanwhile, we observe that these two problems also arise in the construction of a Morita equivalence for more general integrable actions [18, Section 6].

Proof of Lemma 2.5. We first show that $\langle a \cdot x, a \cdot x \rangle_D \leqslant ||a||^2 \langle x, x \rangle_D$ as elements of D, so that $\mu(a) : x \mapsto a \cdot x$ is bounded on $(X_0)_D$. To do this, we again choose a faithful nondegenerate representation π of B, and it is enough to prove that

$$(\bar{\pi}(||a||^2 \langle x, x \rangle_D - \langle a \cdot x, a \cdot x \rangle_D)\pi(b)h \mid \pi(b)h) \geqslant 0 \tag{2.2}$$

for all $b \in B_0$ and $h \in \mathscr{H}_\pi$. We know that

$$||a||^2 \langle x, x \rangle_B - \langle a \cdot x, a \cdot x \rangle_B$$

is positive in B, so

$$(\pi(b^* \beta_s(||a||^2 \langle x, x \rangle_B - \langle a \cdot x, a \cdot x \rangle_B)b)h \mid h) \geqslant 0$$

for all b, h and s. Integrating this over G and pulling the integral inside the inner product gives

$$\left(\pi\left(\int_G b^* \beta_s(||a||^2 \langle x, x \rangle_B - \langle a \cdot x, a \cdot x \rangle_B)b \, ds\right)h \;\middle|\; h\right) \geqslant 0,$$

which is (2.2). We deduce that $\mu(a)$ is bounded. Since a^* is just another element of A_0, $\mu(a^*)$ is also bounded. We can therefore show that $\mu(a)$ is adjointable with adjoint $\mu(a^*)$ by checking that

$$b\langle a\cdot x, y\rangle_D = b\langle x, a^*\cdot y\rangle_D \quad \text{for } b\in B_0,$$

and this follows easily from $\langle a\cdot x, y\rangle_B = \langle x, a^*\cdot y\rangle_B$. Thus $\mu(a)\in\mathscr{L}(\overline{(X_0)_D})$, and it is easy to check that μ is a homomorphism of C^*-algebras.

To verify that U_s is unitary, we let $b\in B_0$, $x, y\in X_0$ and calculate

$$\begin{aligned} b\langle U_s x, U_s y\rangle_D &= \int_G b\beta_t(\langle \Delta(s)^{1/2}\gamma_s(x), \Delta(s)^{1/2}\gamma_s(y)\rangle_B)\,dt \\ &= \int_G b\beta_{ts}(\langle x, y\rangle_B)\Delta(s)\,dt \\ &= \int_G b\beta_r(\langle x, y\rangle_B)\,dr \\ &= b\langle x, y\rangle_D. \end{aligned}$$

This calculation shows that U_s is bounded with $\|U_s\| = 1$, as is $U_{s^{-1}}$, and that

$$b\langle U_s x, y\rangle_D = b\langle U_s x, U_s(U_{s^{-1}}y)\rangle_D = b\langle x, U_{s^{-1}}y\rangle_D,$$

so U_s is adjointable with $U_s^* = U_{s^{-1}}$. Since we trivially have $U_sU_t = U_{st}$ on X_0, U is a homomorphism into $U\mathscr{L}(\overline{(X_0)_D})$, as claimed.

For $x\in X_0$, the covariance condition $\mu(\alpha_s(a))x = U_s\mu(a)U_s^*x$ follows easily from the formulas and the identity $U_s^* = U_{s^{-1}}$, and it then extends by continuity to $x\in\overline{(X_0)_D}$. $\square$

As we observed in Remark 2.6, we do not know whether (μ, U) is always a covariant representation in $\mathscr{L}(\overline{(X_0)_D}) = M(\mathscr{K}(\overline{(X_0)_D}))$, but we can make it into one by representing $\mathscr{L}(\overline{(X_0)_D})$ on Hilbert space. As usual, we start with a nondegenerate faithful representation π of B. Then $\bar\pi$ is faithful on D, and hence $X_0\text{-Ind}_D^{\mathscr{L}}\bar\pi$ is a faithful representation of $\mathscr{L}(\overline{(X_0)_D})$ on $X_0\otimes_D \mathscr{H}_\pi$. We write (v, V) for the pair $((X_0\text{-Ind}_D^{\mathscr{L}}\bar\pi)\circ\mu, (X_0\text{-Ind}_D^{\mathscr{L}}\bar\pi)\circ U)$, so that

$$v(a)(x\otimes_D h) = (a\cdot x)\otimes_D h \quad\text{and}\quad V_s(x\otimes_D h) = \Delta(s)^{1/2}\gamma_s(x)\otimes_D h$$

for $x\in X_0$ and $a\in A_0$. To see that (v, V) is covariant, we relate it to the right-regular representation $((X\text{-Ind}_B^A\pi)^\sim, \rho)$ of (A, α) on $L^2(G, x\otimes_B \mathscr{H}_\pi)$, which is given by

$$(X\text{-Ind}_B^A\pi)^\sim(a)(\xi)(s) = X\text{-Ind}_B^A\pi(\alpha_s(a))(\xi(s)) \quad\text{and}$$

$$\rho_t(\xi)(s) = \Delta(t)^{1/2}\xi(st).$$

The next lemma is similar to [6, Theorem 1].

Lemma 2.7. *Let π be a faithful nondegenerate representation of B on $\mathscr{H}_\pi$. There is an isometry W of $X_0 \otimes_D \mathscr{H}_\pi$ into $L^2(G, x \otimes_B \mathscr{H}_\pi)$, such that*

$$W(x \otimes_D \pi(b)h)(s) = \gamma_s(x) \otimes_B \pi(b)h \quad \text{for } x \in X_0,\ b \in B_0,$$

and then $W(v, V)W^*$ is the restriction of the regular representation $((X\text{-Ind}_B^A \pi)^\sim, \rho)$ to the range of W. In particular, (v, V) is a covariant representation of (A, α).

Proof. We begin by noting that if $x \in X_0$ and $b \in B_0$, then

$$\begin{aligned}\int_G \|\gamma_s(x) \cdot b\|^2\, ds &= \int_G \|\langle \gamma_s(x) \cdot b, \gamma_s(x) \cdot b \rangle_B\|\, ds \\ &= \int_G \|b^* \beta_s(\langle x, x \rangle_B) b\|\, ds \\ &\leqslant \|b\| \int_G \|b^* \beta_s(\langle x, x \rangle_B)\|\, ds \\ &= \|b\| \int_G \|\langle \gamma_s(x) \cdot b, \gamma_s(x) \rangle_B\|\, ds \\ &\leqslant \|b\|\, \|x\| \int_G \|\gamma_s(x) \cdot b\|\, ds,\end{aligned}$$

which is finite by Definition 2.1(2). Since $\|\gamma_s(x) \otimes_B \pi(b)h\| \leqslant \|\gamma_s(x) \cdot b\|\, \|h\|$, it follows that W maps $X_0 \odot \pi(B_0)\mathscr{H}_\pi$ into $L^2(G, x \otimes_B \mathscr{H}_\pi)$. To see that W is isometric, we fix two vectors $x \otimes_D \pi(b)h$ and $y \otimes_D \pi(c)k$ in $X_0 \odot \pi(B_0)\mathscr{H}_\pi$, and compute

$$\begin{aligned}&(W(x \otimes_D \pi(b)h) \mid W(y \otimes_D \pi(c)k)) \\ &\quad = \int_G (W(x \otimes_D \pi(b)h)(s) \mid W(y \otimes_D \pi(c)k)(s))\, ds \\ &\quad = \int_G (\gamma_s(x) \otimes_B \pi(b)h \mid \gamma_s(y) \otimes_B \pi(c)k)\, ds \\ &\quad = \int_G (\pi(c^* \langle \gamma_s(y), \gamma_s(x) \rangle_B\, b)h \mid k)\, ds \\ &\quad = \left(\pi\left(\int_G c^* \beta_s(\langle y, x \rangle_B) b\, ds \right) h \;\middle|\; k \right) \\ &\quad = (\pi(\langle y, x \rangle_D)\pi(b)h \mid \pi(c)k) \\ &\quad = (x \otimes_D \pi(b)h \mid y \otimes_D \pi(c)k).\end{aligned}$$

Thus, W extends to an isometry on $X_0 \otimes_D \mathscr{H}_\pi = \overline{X_0 \odot \pi(B_0)\mathscr{H}_\pi}$.

Now for $x \otimes_D h \in X_0 \odot \pi(B_0)\mathcal{H}_\pi$, we have

$$\begin{aligned}(Wv(a)(x \otimes_D h))(s) &= W(a \cdot x \otimes_D h)(s) = \gamma_s(a \cdot x) \otimes_B h \\ &= \alpha_s(a) \cdot \gamma_s(x) \otimes_B h = X\text{-Ind}_B^A \pi(\alpha_s(a))(W(x \otimes_D h)(s)) \\ &= ((X\text{-Ind}_B^A \pi)^\sim (a) W(x \otimes_D h))(s),\end{aligned}$$

which proves the first intertwining relation. To check that $WV_s(x \otimes_D h) = \rho_s W(x \otimes_D h)$ is even easier. □

Remark 2.8. It is crucial in this argument that we start with a representation π of B, and the lemma suggests that this choice is one of the reasons that the reduced crossed product is appropriate on the left-hand side. The importance of this issue is emphasized by Example 2.1 of [17], where $B = \mathcal{K}(L^2(G))$ and D_0 is contained in the subalgebra $\rho(C_c(G))$ of $M(B) = B(L^2(G))$. Any representation U of G gives a representation of $C_c(G) \cong \rho(C_c(G))$, which extends to a representation of $\mathcal{K}(L^2(G)) \cong C_0(G) \rtimes G$ precisely when there is a compatible representation of $C_0(G)$; by the imprimitivity theorem, this happens precisely when U is induced from a representation of $\{e\}$, and hence is a regular representation of G.

Lemma 2.9. *For $x, y \in X_0$, define ${}_E\langle x, y\rangle : G \to A_0$ by*

$${}_E\langle x, y\rangle(s) = \Delta(s)^{-1/2}\ {}_A\langle x, \gamma_s(y)\rangle,$$

which belongs to $L^1(G, A)$ by Definition 2.1(1). *Let $\theta_{x,y}$ be the compact operator on $(X_0)_D$ defined by $\theta_{x,y} z = x \cdot \langle y, z\rangle_D$. Then*

$$v \rtimes V({}_E\langle x, y\rangle) = X_0\text{-Ind}_D^{\mathcal{L}} \bar{\pi}(\theta_{x,y}). \tag{2.3}$$

Proof. For $z \in X_0$, $h \in \mathcal{H}_\pi$ and $c \in B_0$, we have

$$\begin{aligned} & v \rtimes V({}_E\langle x, y\rangle)(z \otimes_D \pi(c)h) \\ &= \int_G v(\Delta(s)^{-1/2}\ {}_A\langle x, \gamma_s(y)\rangle)\Delta(s)^{1/2}(\gamma_s(z) \otimes_D \pi(c)h)\, ds \\ &= \int_G ({}_A\langle x, \gamma_s(y)\rangle \cdot \gamma_s(z)) \otimes_D \pi(c)h\, ds \\ &= \int_G (x \cdot \langle \gamma_s(y), \gamma_s(z)\rangle_B) \otimes_D \pi(c)h\, ds \\ &= \int_G (x \cdot \beta_s(\langle y, z\rangle_B)) \otimes_D \pi(c)h\, ds. \end{aligned} \tag{2.4}$$

At this stage we want to whip the ds past $\otimes_D\ \pi(c)h$: then the integral would be that defining $x \cdot \langle y, z\rangle_D = \theta_{x,y}(z)$, and we would be done. Unfortunately, the resulting integral converges in the norm coming from the B-valued inner product, so we have to work to pull it through a balanced tensor product defined using the D-valued inner product.

The $(X_0 \otimes_D \mathscr{H}_\pi)$-valued integral in (2.4) is characterized by its inner products with vectors of the form $w \otimes_D \pi(b)k$ for $b \in B_0$:

$$\begin{aligned}
&\left(\int_G (x \cdot \beta_s(\langle y, z\rangle_B)) \otimes_D \pi(c)h\, ds \mid w \otimes_D \pi(b)k\right) \\
&\quad = \int_G ((x \cdot \beta_s(\langle y, z\rangle_B)) \otimes_D \pi(c)h \mid w \otimes_D \pi(b)k)\, ds \\
&\quad = \int_G (\pi(b^* \langle w, x \cdot \beta_s(\langle y, z\rangle_B)\rangle_D c)h \mid k)\, ds \\
&\quad = \int_G \left(\pi\left(\int_G b^* \beta_t(\langle w, x \cdot \beta_s(\langle y, z\rangle_B)\rangle_B) c\, dt\right) h \,\middle|\, k\right) ds,
\end{aligned}$$

which, because the inner integral is that of a B-valued function, is just

$$\begin{aligned}
&= \int_G \int_G (\pi(b^* \beta_t(\langle w, x \cdot \beta_s(\langle y, z\rangle_B)\rangle_B) c)h \mid k)\, dt\, ds \\
&= \int_G \int_G (\pi(b^* \beta_t(\langle w, x\rangle_B) \beta_{ts}(\langle y, z\rangle_B) c)h \mid k)\, dt\, ds. \qquad (2.5)
\end{aligned}$$

The two elements b and c of B_0 are there to ensure that the integrand in this double integral is integrable on $G \times G$, so that we can apply Fubini's Theorem to continue:

$$\begin{aligned}
(2.5) &= \int_G \int_G (\pi(b^* \beta_t(\langle w, x\rangle_B) \beta_{ts}(\langle y, z\rangle_B) c)h \mid k)\, ds\, dt \\
&= \int_G \int_G (\pi(b^* \beta_t(\langle w, x\rangle_B \beta_s(\langle y, z\rangle_B) c))h \mid k)\, ds\, dt \\
&= \int_G \int_G (\pi(b^* \beta_t(\langle w, x\rangle_B \beta_s(\langle y, z\rangle_B) c))h \mid k)\, dt\, ds. \qquad (2.6)
\end{aligned}$$

We can now go backwards through the previous analysis to see that

$$\begin{aligned}
(2.6) &= (\theta_{x,y}(z) \otimes_D \pi(c)h \mid w \otimes_D \pi(b)k) \\
&= (X_0\text{-Ind}_D^{\mathscr{L}} \bar{\pi}(\theta_{x,y})(z \otimes_D \pi(c)h) \mid w \otimes_D \pi(b)k),
\end{aligned}$$

and the result follows. □

To get our Morita equivalence, we consider

$$E_0 := \mathrm{sp}\{{}_E\langle x, y\rangle : x, y \in X_0\} \subset L^1(G, A).$$

Because we know from Lemma 2.7 that the representation (v, V) is equivalent to a subrepresentation of the regular representation, $v \rtimes V$ extends to a representation of the reduced crossed product $A \rtimes_{\alpha,r} G$. Eq. (2.3) therefore implies that $v \rtimes V$ carries the closure E of E_0 in $A \rtimes_{\alpha,r} G$ into the image $X_0\text{-}\mathrm{Ind}_D^{\mathscr{L}} \bar{\pi}(\mathscr{K}(\overline{(X_0)_D}))$ of the imprimitivity algebra

$$\mathscr{K}(\overline{(X_0)_D}) := \overline{\mathrm{sp}}\{\theta_{x,y} : x, y \in X_0\} \subset \mathscr{L}(\overline{(X_0)_D}).$$

We would *like* to prove next that $v \rtimes V$ is isometric for the reduced norm on E_0. If we could do this, we could deduce that $v \rtimes V$ is an isometric linear isomorphism of E onto $\mathscr{K}(\overline{(X_0)_D})$; this would imply both that E is a C^*-algebra (because $\mathscr{K}$ is, and $v \rtimes V$ is a $*$-algebra homomorphism on $A \rtimes_{\alpha,r} G$), and that $\overline{(X_0)_D}$ is an E–D imprimitivity bimodule.

The program of the previous paragraph works without any problems when the isometry W of Lemma 2.7 maps $X_0 \otimes_D \mathscr{H}_\pi$ onto $L^2(G, x \otimes_B \mathscr{H}_\pi)$, or, equivalently, when

$$\mathrm{sp}\{s \mapsto \gamma_s(x) \otimes_D \pi(b)h : x \in X_0,\ b \in B_0,\ h \in \mathscr{H}_\pi\} \tag{2.7}$$

is dense in $L^2(G, x \otimes_B \mathscr{H}_\pi)$. To make this independent of the choice of Hilbert space, we could ask instead that the corresponding map of $X_0 \otimes_D B$ into $L^2(G, X_B)$ be an isomorphism of Hilbert B-modules. In the examples where X_0, A_0 and B_0 consist of functions of compact support, this seems remarkably similar to asking that the maps $s \mapsto {}_E\langle x, y\rangle(s)$ span a dense subspace of $L^1(G, A)$. So what we have proved at this stage is already potentially interesting:

Proposition 2.10. *Suppose that the functions $\{s \mapsto \gamma_s(x) \cdot b : x \in X_0, b \in B_0\}$ span a dense subspace of $L^2(G, X_B)$, and that the space E_0 is dense in $L^1(G, A)$. Then $\overline{(X_0)_D}$ is an $A \rtimes_{\alpha,r} G$–D imprimitivity bimodule.*

Unfortunately, for arbitrary proper actions we do not see how to prove directly that $v \rtimes V$ is isometric on E. So to establish the Morita equivalence of E and D, we prove that X_0 is an E_0–D_0 pre-imprimitivity bimodule. Since we already know that X_0 is a pre-Hilbert D-module, it remains to show that E_0 is a $*$-algebra which acts by bounded operators on $(X_0)_D$ according to the formula ${}_E\langle x, y\rangle \cdot z = x \cdot \langle y, z\rangle_D$, that ${}_E\langle \cdot, \cdot \rangle$ is then a pre-inner product with respect to the completion E of E_0 (that is, the closure E of E_0 in the reduced norm), and that D acts by bounded operators on ${}_E(X_0)$.

Lemma 2.11. *The pairing ${}_E\langle \cdot, \cdot \rangle$ on X_0 satisfies*

$${}_E\langle x, y\rangle * {}_E\langle z, w\rangle = {}_E\langle x \cdot \langle y, z\rangle_D, w\rangle \quad \textit{and} \quad {}_E\langle x, y\rangle^* = {}_E\langle y, x\rangle,$$

where the product and adjoint are those of $L^1(G, A) \subset A \rtimes_{\alpha,r} G$.

Proof. For the first identity, we compute

$$
\begin{aligned}
{}_E\langle x,y\rangle * {}_E\langle z,w\rangle(s) &= \int_G \Delta(r)^{-1/2}\ {}_A\langle x,\gamma_r(y)\rangle \alpha_r(\Delta(r^{-1}s)^{-1/2}\\
&\quad {}_A\langle z,\gamma_{r^{-1}s}(w)\rangle)\,dr\\
&= \int_G\ {}_A\langle x,\gamma_r(y)\rangle\ {}_A\langle \gamma_r(z),\gamma_s(w)\rangle\,dr\,\Delta(s)^{-1/2}\\
&= \int_G\ {}_A\langle\ {}_A\langle x,\gamma_r(y)\rangle\cdot\gamma_r(z),\gamma_s(w)\rangle\,dr\,\Delta(s)^{-1/2}\\
&= {}_A\left\langle \int_G x\cdot\langle\gamma_r(y),\gamma_r(z)\rangle_B\,dr,\ \gamma_s(w)\right\rangle\ \Delta(s)^{-1/2}\\
&= {}_A\langle x\cdot\langle y,z\rangle_D,\gamma_s(w)\rangle\Delta(s)^{-1/2}\\
&= {}_E\langle x\cdot\langle y,z\rangle_D,w\rangle(s).
\end{aligned}
$$

The second identity follows from a simple algebraic manipulation. □

Proposition 2.12. *The set* $E_0 := \mathrm{sp}\{{}_E\langle x,y\rangle : x,y\in X_0\}$ *is a* $*$*-subalgebra of* $L^1(G,A)$*, and there is a left action of* E_0 *on* $(X_0)_D$ *such that*

$$
{}_E\langle x,y\rangle\cdot z = x\cdot\langle y,z\rangle_D \quad \textit{and}
$$

$$
\langle e\cdot x,e\cdot x\rangle_D \leqslant \|e\|^2\langle x,x\rangle_D \ \textit{as elements of}\ D
$$

for $x,y,z\in X_0$ *and* $e\in E_0$.

Proof. Because $x\cdot\langle y,z\rangle_D$ belongs to X_0, the formulas in Lemma 2.11 show that E_0 is a $*$-subalgebra of $L^1(G,A)$. We know from Lemma 2.9 that the $*$-homomorphism $\phi := (\mathrm{Ind}_D^{\mathscr{L}}\bar{\pi})^{-1}\circ(v\rtimes V)$ restricts to a $*$-homomorphism of E_0 into $\mathscr{L}(\overline{(X_0)_D})$ such that $\phi({}_E\langle x,y\rangle) = \theta_{x,y}$, and ϕ gives the required action of E_0 on $(X_0)_D$: $e\cdot z$ is by definition $\phi(e)z$. For any bounded operator $T\in\mathscr{L}(\overline{(X_0)_D})$, we have $\langle Tx,Tx\rangle_D \leqslant \|T\|^2\langle x,x\rangle_D$; because $\mathrm{Ind}_D^{\mathscr{L}}\bar{\pi}$ is isometric and $v\rtimes V$ is decreasing for the reduced norm, we have $\|\phi(e)\|\leqslant\|e\|$, and the inequality follows. □

Proposition 2.13. *The pairing* ${}_E\langle\cdot,\cdot\rangle$ *is an inner product with values in the* C^**-algebra* $E := \overline{E_0}$*, and*

$$
{}_E\langle x\cdot d,x\cdot d\rangle \leqslant \|d\|^2\ {}_E\langle x,x\rangle \quad \textit{as elements of the } C^*\textit{-algebra } E. \tag{2.8}
$$

Proof. We know from Lemma 2.11 that

$$
{}_E\langle x,y\rangle *\ {}_E\langle z,w\rangle =\ {}_E\langle x\cdot\langle y,z\rangle_D,w\rangle;
$$

since the left action of E_0 satisfies ${}_E\langle x, y\rangle \cdot z = x \cdot \langle y, z\rangle_D$, it follows that $e \cdot {}_E\langle z, w\rangle = {}_E\langle e \cdot z, w\rangle$ for all $e \in E_0$. Lemma 2.11 also shows that ${}_E\langle x, y\rangle^* = {}_E\langle y, x\rangle$, so ${}_E\langle \cdot, \cdot\rangle$ has the required algebraic properties.

To see positivity, we fix a representation ρ of A on $\mathscr{H}$, and consider the left-regular representation $(\tilde{\rho}, \lambda)$ on $L^2(G, \mathscr{H})$ given by

$$(\tilde{\rho}(a)\xi)(s) = \rho(\alpha_{s^{-1}}(a))\xi(s) \quad \text{and} \quad (\lambda_t\xi)(s) = \lambda(t^{-1}s).$$

Now we let $x \in X_0$ and consider $\xi \in L^2(G, \mathscr{H})$ of the form $s \mapsto f(s)h$ for $f \in C_c(G)^+$. Then, we have

$$\begin{aligned}
&(\tilde{\rho} \rtimes \lambda({}_E\langle x, x\rangle)\xi \mid \xi) \\
&= \left(\int_G \tilde{\rho}({}_E\langle x, x\rangle(s))\lambda_s\xi \, ds \mid \xi \right) \\
&= \int_G \left(\left(\int_G \tilde{\rho}(\Delta(s)^{-1/2}{}_A\langle x, \gamma_s(x)\rangle \lambda_s\xi \, ds \right)(t) \;\middle|\; \xi(t) \right) dt \\
&= \int_G \int_G (\rho(\alpha_{t^{-1}}({}_A\langle x, \gamma_s(x)\rangle))(\xi(s^{-1}t)) \mid \xi(t))\Delta(s)^{-1/2} \, ds \, dt \\
&= \int_G \int_G (\rho({}_A\langle \gamma_{t^{-1}}(x)f(t), \gamma_{t^{-1}s}(x)f(s^{-1}t)\rangle)h \mid h)\Delta(s)^{-1/2} \, ds \, dt.
\end{aligned}$$

Since the function $t \mapsto \gamma_{t^{-1}}(x)f(t)$ belongs to $C_c(G, X)$, we can apply Fubini's Theorem, and then substitute $r = s^{-1}t$ to reduce this to

$$\int_G \int_G (\rho({}_A\langle \gamma_{t^{-1}}(x)f(t)\Delta(t)^{-1/2}, \gamma_{r^{-1}}(x)f(r)\Delta(r)^{-1/2}\rangle)h \mid h) \, dr \, dt,$$

which has the form $(\rho({}_A\langle y, y\rangle)h \mid h)$ for the element y of X defined by

$$y := \int_G \gamma_{t^{-1}}(x)f(t)\Delta(t)^{-1/2} \, dt,$$

and hence is positive. Since ξ of the given form span a dense subspace of $L^2(G, \mathscr{H})$, this proves that $\tilde{\rho} \rtimes \lambda({}_E\langle x, x\rangle)$ is positive.

Thus ${}_E\langle \cdot, \cdot\rangle$ is a pre-inner product; it is definite because $\|{}_E\langle x, x\rangle\|_{A \rtimes_r G} = 0$ implies ${}_E\langle x, x\rangle(s) = 0$ for all $s \in G$. If now $d \in D \subset M(B)^\beta$, then $\gamma_s(x \cdot d) = \gamma_s(x) \cdot \beta_s(d) = \gamma_s(x) \cdot d$, so we can repeat the calculation of the previous paragraph to see that

$$\begin{aligned}
&(\tilde{\rho} \rtimes \lambda(\|d\|^2{}_E\langle x, x\rangle - {}_E\langle x \cdot d, x \cdot d\rangle)\xi \mid \xi) \\
&= (\rho(\|d\|^2{}_A\langle y, y\rangle - {}_A\langle y \cdot d, y \cdot d\rangle)h \mid h),
\end{aligned}$$

which is positive because $M(B)$ acts as bounded operators on ${}_AX$. This proves (2.8). □

Remark 2.14. As in [17], E is an ideal in $A \rtimes_{\alpha,r} G$. To see this, one observes first that the dense subalgebra A_0 of A satisfies $A_0 E_0 \subset E_0$ and second that $s \mapsto \alpha_t(e(t^{-1}s))$ is in E_0 for every $e \in E_0$ and $t \in G$. So if f is in the dense subalgebra $C_c(G, A_0)$ of $A \rtimes_{\alpha,r} G$, then the integrand of $f * e(s) = \int_G f(t)\alpha_t(e(t^{-1}s))\,dt$ is in E_0 for each t, and hence the integral takes values in $\overline{E_0} = E$. Since E_0 is a $*$-subalgebra this implies that E is a two-sided ideal.

Definition 2.15. Following [17], we say that a proper action γ on ${}_A X_B$ is *saturated* with respect to X_0 if $E = A \rtimes_{\alpha,r} G$.

To sum up, we have now proved:

Theorem 2.16. *Suppose the Morita equivalence* ${}_{(A,\alpha)}(X, \gamma)_{(B,\beta)}$ *is proper in the sense of Definition* 2.1 *with respect to the pre-imprimitivity bimodule* ${}_{A_0}(X_0)_{B_0}$. *Then the completion of* X_0 *in the norm* $\|x\| \coloneqq \|\langle x, x\rangle_D\|^{1/2}$ *implements a Morita equivalence between the ideal*

$$E \coloneqq \overline{\mathrm{sp}}\{s \mapsto \Delta(s)^{-1/2}\ {}_A\langle x, \gamma_s(y)\rangle : x, y \in X_0\} \subset A \rtimes_{\alpha,r} G$$

and the C^**-algebra*

$$D \coloneqq \overline{\mathrm{sp}}\{\langle x, y\rangle_D : x, y \in X_0\} \subset M(B).$$

In particular, if the action γ *is saturated, then* $\overline{(X_0)_D}$ *implements a Morita equivalence between* $A \rtimes_{\alpha,r} G$ *and* D.

The general theory produces a pre-imprimitivity bimodule on which only the spans of the ranges of the inner products act. In the main examples, there are algebras of continuous functions of compact support which ought to act too, and it is important that the formulas extend. The following lemma gives conditions under which the extended left action is given by the expected formula.

Lemma 2.17. *Let* $f : G \to A$ *be a continuous function such that both* $s \mapsto \|f(s)\|$ *and* $s \mapsto \|f(s)\|\Delta(s)^{1/2}$ *are integrable. Suppose that* $x \in X_0$ *and that the integral* $\int_G f(s) \cdot \gamma_s(x)\Delta(s)^{1/2}\,ds$, *which converges in* X_B *because of our second integrability hypothesis on* f, *belongs to* X_0. *Let* π *be a representation of* B *and let* (v, V) *be the covariant representation discussed in Lemma* 2.7; *note that the first integrability hypothesis on* f *implies that* $v \rtimes V(f)$ *makes sense as a bounded operator on* $X_0 \otimes_D \mathscr{H}_\pi$. *Then*

$$\left(\int_G f(s) \cdot \gamma_s(x)\Delta(s)^{1/2}\,ds\right) \otimes_D h = v \rtimes V(f)(x \otimes_D h) \quad \textit{for } h \in \mathscr{H}_\pi.$$

In other words, the left action of $f \in L^1(G,A)$ *on* $x \in (X_0)_D$ *is given by the integral formula*

$$f \cdot x = \int_G f(s) \cdot \gamma_s(x) \Delta(s)^{1/2} \, ds.$$

Proof. To make things easier on the eye, we write $g(s) := f(s) \cdot \gamma_s(x)\Delta(s)^{1/2}$. We fix $y \otimes_D \pi(b)k \in X_0 \otimes_D \pi(B_0)\mathscr{H}_\pi$, and compute:

$$\begin{aligned}
&(v \rtimes V(f)(x \otimes_D h) \mid y \otimes_D \pi(b)k) \\
&\quad = \left(\int_G v(f(s)) V_s(x \otimes_D h) \, ds \,\middle|\, y \otimes_D \pi(b)k \right) \\
&\quad = \left(\int_G (g(s) \otimes_D h) \, ds \,\middle|\, y \otimes_D \pi(b)k \right) \\
&\quad = \int_G (g(s) \otimes_D h \mid y \otimes_D \pi(b)k) \, ds \\
&\quad = \int_G (\pi(b^* \langle y, g(s) \rangle_D) h \mid k) \, ds \\
&\quad = \int_G \left(\pi\left(\int_G b^* \beta_t(\langle y, g(s) \rangle_B) \, dt \right) h \mid k \right) ds \\
&\quad = \int_G \int_G (\pi(b^* \beta_t(\langle y, g(s) \rangle_B)) h \mid k) \, dt \, ds,
\end{aligned} \tag{2.9}$$

using standard properties of B-valued integrals. On the other hand,

$$\begin{aligned}
&\left(\left(\int_G g(s) \, ds \right) \otimes_D h \mid y \otimes_D \pi(b)k \right) \\
&\quad = \left(\pi\left(b^* \left\langle y, \int_G g(s) \, ds \right\rangle_D \right) h \mid k \right) \\
&\quad = \left(\pi\left(\int_G b^* \beta_t\left(\left\langle y, \int_G g(s) \, ds \right\rangle_B \right) dt \right) h \mid k \right) \\
&\quad = \int_G \left(\pi\left(b^* \beta_t\left(\left\langle y, \int_G g(s) \, ds \right\rangle_B \right) \right) h \mid k \right) dt.
\end{aligned}$$

Because the inside integral converges in X_B, we can pull it through the B-valued inner product with y; now we have an ordinary B-valued integral, and we can pull the automorphisms and representation through to recover

$$\int_G \int_G (\pi(b^* \beta_t(\langle y, g(s) \rangle_B)) h \mid k) \, ds \, dt. \tag{2.10}$$

But now we are talking about ordinary scalar-valued integrals; the element b is a sum of elements of the form $\langle w, z\rangle_B$, and for such an element the integrand

$$(\pi(b^*\beta_t(\langle y, g(s)\rangle_B))h \mid k) = (\pi(\langle {}_A\langle \gamma_t(y), w\rangle \cdot z, \gamma_t(g(s))\rangle_B))h \mid k)$$

is integrable on $G \times G$. Thus, an application of Fubini's Theorem identifies (2.10) with (2.9) and we are done. □

3. Tensor-product decompositions of imprimitivity bimodules

Suppose the Morita equivalence ${}_{(A,\alpha)}(X,\gamma)_{(B,\beta)}$ is proper with respect to the pre-imprimitivity module ${}_{A_0}(X_0)_{B_0}$, so that Theorem 2.16 gives a Morita equivalence $\overline{(X_0)}$ between an ideal in $A \rtimes_{\alpha,r} G$ and a generalized fixed-point algebra D for (B,β). We now want to show that (B,β) is itself proper in Rieffel's sense [17, Definition 1.2], and to relate $\overline{(X_0)}$ to other Morita equivalences involving Rieffel's generalized fixed-point algebra B^β. Both D and B^β are by definition subalgebras of $M(B)^\beta$, but it is not obvious that they must be the same subalgebra. Indeed, it is not even obvious that we get the same generalized fixed-point algebra when β is proper with respect to two different dense $*$-subalgebras. Fortunately, we have been able to show that at least when γ is saturated, all the fixed-point algebras relevant to us coincide. After we have sorted this out, it will be relatively easy to get the desired relations between imprimitivity bimodules.

We begin by giving a careful statement of our main results. Since we are concerned about possibly different fixed-point algebras, we shall denote by $(B,B_1)^\beta$ the generalized fixed-point algebra as defined in [17] when β is proper with respect to a particular subalgebra B_1.

Theorem 3.1. *Suppose that the Morita equivalence ${}_{(A,\alpha)}(X,\gamma)_{(B,\beta)}$ is proper with respect to ${}_{A_0}(X_0)_{B_0}$, with generalized fixed-point algebra D. Then*

(1) *the action β on B is proper with respect to the subalgebra*

$$B_1 := \langle X_0, X_0\rangle_B := \operatorname{sp}\{\langle x, y\rangle_B : x, y \in X_0\},$$

and the generalized fixed-point algebra $(B,B_1)^\beta$ is an ideal in D;

(2) *the action γ is also proper with respect to the smaller pre-imprimitivity bimodule ${}_{A_0}(X_0)_{B_1}$, and then has the same generalized fixed-point algebra D.*

The action β is saturated with respect to B_1 if and only if γ is saturated with respect to X_0, and then $(B,B_1)^\beta = D$. There is an isomorphism

$$\Omega : (X \rtimes_r G) \otimes_{B \rtimes_r G} \overline{B_1} \to \overline{X_0}$$

of $(A \rtimes_{\alpha,r} G)$*–D-imprimitivity bimodules such that*

$$\Omega(f \otimes b) = \int_G f(s) \cdot \beta_s(b) \Delta(s)^{1/2}\, ds \tag{3.1}$$

for $b \in B_1$ *and* f *of the form* $f(s) := x \cdot \beta_s(c^*) \Delta(s)^{-1/2}$ *with* $x \in X_0$ *and* $c \in B_1$.

Proofs of (1) and (2). For (1), we need to verify the two items of [17, Definition 1.2]. Write F for $(B, B_1)^\beta$, and let $b = \langle v, w \rangle_B$ and $c = \langle x, y \rangle_B$ be typical spanning elements of B_1. Then

$$b\beta_s(c^*) = \langle v, w \cdot \beta_s(\langle y, x \rangle_B) \rangle_B = \langle v, \; {}_A\langle w, \gamma_s(y) \rangle \cdot \gamma_s(x) \rangle_B,$$

and hence

$$\|b\beta_s(c^*)\| \leqslant \|v\| \|x\| \|{}_A\langle w, \gamma_s(y) \rangle\|.$$

The function $s \mapsto {}_A\langle w, \gamma_s(y) \rangle$ and its product with $s \mapsto \Delta(s)^{1/2}$ are in $L^1(G, A)$ because γ is proper, so it follows that $s \mapsto b\beta_s(c)$ and $s \mapsto \Delta(s)^{1/2} b\beta_s(c)$ are integrable. This gives the first item of [17, Definition 1.2].

Set $z := v \cdot \langle x, y \rangle_B$. Then $z \in X_0$, and

$$b^* c = \langle w, v \rangle_B \langle x, y \rangle_B = \langle w, v \cdot \langle x, y \rangle_B \rangle_B = \langle w, z \rangle_B. \tag{3.2}$$

Definition 2.1(3) says there is a multiplier $\langle w, z \rangle_D \in M(B_0)^\beta$ such that

$$\int_G a\beta_s(b^* c)\, ds \overset{(3.2)}{=} \int_G a\beta_s(\langle w, z \rangle_B)\, ds$$
$$= a \cdot \langle w, z \rangle_D \quad \text{for all } a \in B_1 \subset B_0. \tag{3.3}$$

We claim that $\langle w, z \rangle_D$ multiplies B_1 (we already know it multiplies B_0). If $b' = \langle x_1, x_2 \rangle_B \in B_1$, then $b' \cdot \langle w, z \rangle_D = \langle x_1, x_2 \cdot \langle w, z \rangle_D \rangle_B$ because $\langle w, z \rangle_D \in M(B)$. But $x_2 \cdot \langle w, z \rangle_D$ is back in X_0, so $b' \cdot \langle w, z \rangle_D \in \langle X_0, X_0 \rangle_B = B_1$. Thus $\langle w, z \rangle_D \in M(B_1)$.

We now define $\langle b, c \rangle_F := \langle w, z \rangle_D$, and (3.3) gives the second item of [17, Definition 1.2]. Note that by definition of $\langle \cdot, \cdot \rangle_F$, we have $F = \overline{\langle X_0, X_0 \cdot B_1 \rangle_D}$. Since $X_0 \cdot B_1$ is a sub-module of ${}_{E_0}(X_0)_{D_0}$, it follows from the Rieffel correspondence that F is an ideal of D. This gives (1).

For (2), we have to verify the three properties of Definition 2.1 for ${}_{A_0}(X_0)_{B_1}$. Since $B_1 \subset B_0$, the integrability properties are clear. So it suffices to check Definition 2.1(3) and to show that D and $D_1 := D({}_{A_0}(X_0)_{B_1})$ coincide. If $x, y, z \in X_0$ then $z \cdot \langle x, y \rangle_D$ is in X_0, so

$$\langle w, z \rangle_B \langle x, y \rangle_D = \langle w, z \cdot \langle x, y \rangle_D \rangle_B$$

is in B_1. Thus $\langle x,y\rangle_D$ multiplies B_1, and $\langle x,y\rangle_{D_1} := \langle x,y\rangle_D$ has the properties described in Definition 2.1(3). But with this definition of $\langle\cdot,\cdot\rangle_{D_1}$ we trivially have $D=D_1$. $\square$

The proof of the decomposition isomorphism (3.1) uses a general lemma about imprimitivity bimodules over a linking algebra. If ${}_AX_B$ is an imprimitivity bimodule, we denote by $\tilde{X}$ the dual B–A imprimitivity bimodule of [15, p. 49], and by

$$L(X) := \left\{ \begin{pmatrix} a & x \\ \flat(y) & b \end{pmatrix} : a\in A, b\in B, x,y\in X \right\}$$

the linking algebra of X [15, p. 50]. The matrices

$$p = p_{L(X)} = \begin{pmatrix} 1_{M(A)} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q = q_{L(X)} = \begin{pmatrix} 0 & 0 \\ 0 & 1_{M(B)} \end{pmatrix}$$

define full projections in $M(L(X))$, and the corners $pL(X)p$, $qL(X)q$ and $pL(X)q$ in $L(X)$ can be naturally identified with A, B and X, respectively.

In the next lemma, which is a variation of [3, Lemma 4.6; 4, Proposition 4.3], we use the identifications to produce actions of A, B and X on a module over $L(X)$.

Lemma 3.2. *Let X be an A–B imprimitivity bimodule with linking algebra $L(X)$. If Z is an $L(X)$–C imprimitivity bimodule, then pZ and qZ are A–C and B–C imprimitivity bimodules, respectively, and there is an isomorphism $\Omega : x\otimes_B qZ \to pZ$ of A–C imprimitivity bimodules such that*

$$\Omega(x\otimes_B qz) = x\cdot qz. \tag{3.4}$$

Proof. Since $A = pL(X)p \subset L(X)$, it is easy to see that pZ is an A-module; on pZ, the $L(X)$-valued inner product takes values in $pL(X)p$, and with ${}_A\langle pz,pz'\rangle := p_{L(X)}\langle z,z'\rangle p$, pZ becomes a full left Hilbert A-module. The right actions and inner products are already defined; the only thing we need to worry about is whether pZ is full as a Hilbert C-module. So let I be the ideal in C spanned by the elements $\langle pz,pz'\rangle_C$. Then

$$\begin{aligned} Z\text{-Ind}_C^{L(X)}(I) &= \overline{\text{sp}}\{{}_{L(X)}\langle z\cdot i, z'\rangle : z,z'\in Z, i\in I\} \\ &= \overline{\text{sp}}\{{}_{L(X)}\langle z\cdot \langle pw,pw'\rangle_C, z'\rangle : z,z',w,w'\in Z\} \end{aligned}$$

$$= \overline{\mathrm{sp}}\{_{L(X)}\langle {}_{L(X)}\langle z, pw\rangle \cdot pw', z'\rangle : z, z', w, w' \in Z\}$$

$$= \overline{\mathrm{sp}}\{_{L(X)}\langle z, pw\rangle_{L(X)}\langle pw', z'\rangle : z, z', w, w' \in Z\}$$

$$= \overline{\mathrm{sp}}\{_{L(X)}\langle z, w\rangle p^* p_{L(X)}\langle w', z'\rangle : z, z', w, w' \in Z\}$$

$$= \overline{L(X)pL(X)},$$

which is $L(X)$ because p is full. We can therefore deduce from the Rieffel correspondence that $I = C$. Thus, pZ is an A–C imprimitivity bimodule. Similarly, qZ is a B–C imprimitivity bimodule.

Note that the map $(x, qz) \mapsto x \cdot qz$ is bilinear, so there is a well-defined map Ω on the algebraic tensor product $X \odot qZ$ satisfying (3.4), and which is C-linear. To see that it is A-linear, recall that the action of A on X is given by the product of the embedded copies in $L(X)$; thus for $a \in A$ and $x \in X$, we have

$$\Omega(a \cdot x \otimes qz) = (a \cdot x) \cdot qz = a \cdot (x \cdot qz) = a \cdot \Omega(x \otimes qz).$$

In the same way, the inner product $\langle y, x\rangle_B$ is given by the product $y^* x$ in $L(X)$, so

$$\begin{aligned}\langle x \otimes_B qz, y \otimes_B qw\rangle_C &= \langle \langle y, x\rangle_B \cdot qz, qw\rangle_C = \langle (y^* x) \cdot qz, qw\rangle_C \\ &= \langle x \cdot qz, y \cdot qw\rangle_C \\ &= \langle \Omega(x \otimes_B qz), \Omega(y \otimes_B qw)\rangle_C,\end{aligned}$$

and Ω extends to an isometry of $(x \otimes_B qZ)_C$ into $(pZ)_C$. To see that Ω has a dense range and is therefore onto, note that $L(X)$ acts nondegenerately on Z, so that

$$\begin{aligned}\operatorname{range} \Omega \supset pL(X)q \cdot qZ &= pL(X)q \cdot qL(X) \cdot Z \\ &= pL(X)qL(X) \cdot Z = pL(X) \cdot Z \\ &= pZ\end{aligned}$$

because q is full. Since Ω is a bimodule isomorphism which preserves the C-valued inner product, it must preserve the A-valued inner product as well. □

To prove Theorem 3.1, we apply Lemma 3.2 to the Combes bimodule $X \rtimes_{\gamma, r} G$ and a bimodule Z coming from an application of Theorem 2.16. As it arises, Z will be a left module over $L(X) \rtimes_r G$ rather than $L(X \rtimes_r G)$. Thus, we shall have to identify $L(X) \rtimes_r G$ with $L(X \rtimes_r G)$. Since we need to be very explicit about the identifications involved, we review the details.

Suppose as in the theorem that ${}_{(A,\alpha)}(X,\gamma)_{(B,\beta)}$ is a Morita equivalence. For $f \in C_c(G,A), g \in C_c(G,B)$ and $z, w \in C_c(G,X)$ define

$$f \cdot z(s) = \int_G f(r) \cdot \gamma_r(z(r^{-1}s))\, dr \tag{3.5}$$

$$z \cdot g(s) = \int_G z(r) \cdot \beta_r(g(r^{-1}s))\, dr \tag{3.6}$$

$${}_{A \rtimes_\alpha G}\langle z, w \rangle(s) = \int_G {}_A\langle z(r), \gamma_s(w(s^{-1}r)) \rangle \Delta_G(s^{-1}r)\, dr \tag{3.7}$$

$$\langle z, w \rangle_{B \rtimes_\beta G}(s) = \int_G \beta_r^{-1}(\langle z(r), w(rs) \rangle_B)\, dr. \tag{3.8}$$

Proposition 3.3 (Combes). *With the above actions and inner products, $C_c(G,X)$ is a $C_c(G,A)$–$C_c(G,B)$ pre-imprimitivity bimodule whose completion is an $A \rtimes_{\alpha,r} G$–$B \rtimes_{\beta,r} G$ imprimitivity bimodule $X \rtimes_{\gamma,r} G$.*

Composing functions with the usual identifications of the corners in $L(X)$ gives embeddings $\imath_{11}$, $\imath_{12}$ and $\imath_{22}$ of $C_c(G,A)$, $C_c(G,X)$ and $C_c(G,B)$, respectively, in $C_c(G, L(X))$. The actions of G on the corners combine to give an action u of G on $L(X)$. The following result is proved in [1].

Proposition 3.4 (Combes). *The maps $\imath_{ij}$ induce an isomorphism of $L(X \rtimes_{\gamma,r} G)$ onto $L(X) \rtimes_{u,r} G$ which carries $p_{L(X \rtimes_{\gamma,r} G)}$ and $q_{L(X \rtimes_{\gamma,r} G)}$ into full projections $\hat{p}$ and $\hat{q} \in M(L(X) \rtimes_{u,r} G)$ such that $\hat{p}(L(X) \rtimes_{u,r} G)\hat{p}$, $\hat{q}(L(X) \rtimes_{u,r} G)\hat{q}$ and $\hat{p}(L(X) \rtimes_{u,r} G)\hat{q}$ are identified with $A \rtimes_{\alpha,r} G$, $B \rtimes_{\beta,r} G$ and $X \rtimes_{\gamma,r} G$, respectively.*

We now return to the situation of Theorem 3.1. Recall that we seek an $L(X \rtimes_{\gamma,r} G)$–$D$ imprimitivity bimodule Z to which we can apply Lemma 3.2. We intend to find Z by applying Theorem 2.16 to the Morita equivalence ${}_{(L(X),u)}(X \oplus B)_{(B,\beta)}$ and identifying the left-hand algebra $L(X) \rtimes_{u,r} G$ with $L(X \rtimes_{\gamma,r} G)$ using Proposition 3.4. Of course we have some checking to do:

Lemma 3.5. *Suppose that ${}_{(A,\alpha)}(X,\gamma)_{(B,\beta)}$ is proper with respect to the pre-imprimitivity bimodule ${}_{A_0}(X_0)_{B_0}$, and has generalized fixed-point algebra D. Let $B_1 = \langle X_0, X_0 \rangle_B$ and*

$$L(X_0) := \begin{pmatrix} A_0 & X_0 \\ \tilde{X}_0 & B_1 \end{pmatrix}.$$

Then $\gamma \oplus \beta$ is proper with respect to ${}_{L(X_0)}(X_0 \oplus B_1)_{B_1}$, and has generalized fixed-point algebra D.

Proof. Let $x, y \in X_0$ and $b, c \in B_1$. For Definition 2.1(1) we need to verify that

$$s \mapsto {}_{L(X)}\left\langle \begin{pmatrix} x \\ b \end{pmatrix}, \begin{pmatrix} \gamma_s(y) \\ \beta_s(c) \end{pmatrix} \right\rangle = \begin{pmatrix} {}_A\langle x, \gamma_s(y) \rangle & x \cdot \beta_s(c^*) \\ \flat(\gamma_s(y) \cdot b^*) & b\beta_s(c^*) \end{pmatrix}$$

and its product with $\Delta(s)^{-1/2}$ are in $L^1(G, L(X))$. Since the action of G is proper with respect to ${}_{A_0}(X_0)_{B_0}$ and ${}_{B_1}(B_1)_{B_1}$, the functions $s \mapsto {}_A\langle x, \gamma_s(y) \rangle$ and $s \mapsto b\beta_s(c^*)$ as well as their products with $\Delta(s)^{-1/2}$ are integrable. That $s \mapsto x \cdot \beta_s(c)$ and $s \mapsto x \cdot \beta_s(c)\Delta(s)^{-1/2}$ are integrable follows from the estimate

$$||x \cdot \beta_s(\langle y, z \rangle_B)|| \leqslant || {}_A\langle x, \gamma_s(y) \rangle \cdot \gamma_s(z)|| \leqslant || {}_A\langle x, \gamma_s(y) \rangle|| \, ||z||.$$

For Definition 2.1(2), note that $s \mapsto \gamma_s(x) \cdot b$ and $s \mapsto \beta_s(b) \cdot c$ and their products with $\Delta(s)^{-1/2}$ are integrable using Definition 2.1(2) for ${}_{A_0}(X_0)_{B_0}$ and ${}_{B_1}(B_1, \beta)_{B_1}$.

To verify Definition 2.1(3), we write D' for the generalized fixed-point algebra associated to the action $\gamma \oplus \beta$, and $F := (B, B_1)^\beta$, and define

$$\left\langle \begin{pmatrix} x \\ b \end{pmatrix}, \begin{pmatrix} y \\ c \end{pmatrix} \right\rangle_{D'} := \langle x, y \rangle_D + \langle b, c \rangle_F.$$

Note that D multiplies B_1, and that the right-hand side belongs to D because $F \subset D$ by part (1) of the theorem. Thus $D' = D$. Straightforward calculations show that

$$\begin{pmatrix} z \\ b' \end{pmatrix} \cdot \left\langle \begin{pmatrix} x \\ b \end{pmatrix}, \begin{pmatrix} y \\ c \end{pmatrix} \right\rangle_D \in \begin{pmatrix} X_0 \\ B_1 \end{pmatrix}$$

and that

$$\int_G b'\beta_s\left(\left\langle \begin{pmatrix} x \\ b \end{pmatrix}, \begin{pmatrix} y \\ c \end{pmatrix} \right\rangle_B\right) ds = b'\left\langle \begin{pmatrix} x \\ b \end{pmatrix}, \begin{pmatrix} y \\ c \end{pmatrix} \right\rangle_D$$

for $z \in X_0$ and $b' \in B_1$. □

At this point it is convenient to prove the assertion in Theorem 3.1 about saturated actions.

Proposition 3.6. *Suppose that the Morita equivalence ${}_{(A,\alpha)}(X, \gamma)_{(B,\beta)}$ is proper with respect to the pre-imprimitivity bimodule ${}_{A_0}(X_0)_{B_0}$. Then the following are equivalent*:

(1) *the action γ is saturated with respect to X_0;*
(2) *the action β is saturated with respect to B_1; and*
(3) *the action $\gamma \oplus \beta$ is saturated with respect to $X_0 \oplus B_1$.*

For the proof we need a standard lemma.

Lemma 3.7. *Let E be an ideal in a C^*-algebra C and let p be a full projection in $M(C)$. Then $E = C$ if and only if $pEp = pCp$.*

Proof. Recall that pC is a pCp–C imprimitivity bimodule. The result follows from the Rieffel correspondence:

$$\begin{aligned} pC\text{-Ind}(E) &= \overline{\mathrm{sp}}\{{}_{pCp}\langle pce, pd\rangle : c, d \in C, e \in E\} \\ &= \overline{\mathrm{sp}}\{pced^*p : c, d \in C, e \in E\} \\ &= pEp \end{aligned}$$

is the corresponding ideal in pCp. □

Proof of Proposition 3.6. As usual, we write E for the ideal in $L(X) \rtimes_{u,r} G$ spanned by functions of the form

$$\begin{aligned} s \mapsto \Delta(s)^{-1/2} \; {}_{L(X)}\left\langle \begin{pmatrix} x \\ b \end{pmatrix}, \begin{pmatrix} \gamma_s(y) \\ \beta_s(c) \end{pmatrix} \right\rangle \\ = \Delta(s)^{-1/2} \begin{pmatrix} {}_A\langle x, \gamma_s(y)\rangle & x \cdot \beta_s(c^*) \\ \flat(\gamma_s(y) \cdot b^*) & b\beta_s(c^*) \end{pmatrix}, \end{aligned} \tag{3.9}$$

where $x, y \in X_0$ and $b, c \in B_1$. Since $L(X) \rtimes_{u,r} G \cong L(X \rtimes_{\gamma,r} G)$, two applications of Lemma 3.7 imply that

$$E = L(X) \rtimes_{u,r} G \Longleftrightarrow \hat{p}\, L(X \rtimes_{\gamma,r} G)\hat{p} = A \rtimes_{\alpha,r} G \Longleftrightarrow \hat{q}\, L(X \rtimes_{\gamma,r} G)\hat{q} = B \rtimes_{\beta,r} G.$$

Thus, functions of form (3.9) are dense in $L(X) \rtimes_{u,r} G$ if and only if $A \rtimes_{\alpha,r} G$ is spanned by the functions $s \mapsto \Delta(s)^{-1/2} \; {}_A\langle x, \gamma_s(y)\rangle$ if and only if $B \rtimes_{\beta,r} G$ is spanned by the functions $s \mapsto \Delta(s)^{-1/2} b\beta_s(c^*)$. The result follows. □

Proof of Theorem 3.1. Parts (1) and (2) were proved earlier, and the statement about saturation is part of Proposition 3.6. We know from Lemma 3.5 and Proposition 3.6 that γ and $\gamma \oplus \beta$ are proper and saturated with respect to X_0 and ${}_{L(X_0)}(X_0 \oplus B_1)_{B_1}$, respectively. Thus Theorem 2.16 gives two imprimitivity bimodules

$$_{B \rtimes_{\beta,r} G}(\overline{B_1})_F \quad \text{and} \quad {}_{L(X) \rtimes_{u,r} G}(\overline{X_0 \oplus B_1})_D, \tag{3.10}$$

where F is an ideal of D. Since $L(X) \rtimes_{u,r} G \cong L(X \rtimes_{\gamma,r} G)$, we can apply Lemma 3.2 to the imprimitivity bimodules $X \rtimes_{\gamma,r} G$ and $\overline{X_0 \oplus B_1}$. Thus to see the existence of the isomorphism, it suffices to prove that

$$\hat{p}\,(\overline{X_0 \oplus B_1}) \cong {}_{A \rtimes_{\alpha,r} G}(\overline{X_0})_D \quad \text{and} \quad \hat{q}\,(\overline{X_0 \oplus B_1}) \cong {}_{B \rtimes_{\beta,r} G}(\overline{B_1})_D \tag{3.11}$$

as imprimitivity bimodules; given this, it then follows from the Rieffel correspondence that $F = D$ because the imprimitivity bimodules in (3.10) and (3.11) based on B_1 are completed in the same norm.

That $\hat{p}\,(\overline{X_0 \oplus B_1})$ and $\hat{q}\,(\overline{X_0 \oplus B_1})$ are $(A \rtimes_{\alpha,r} G)$–D and $(B \rtimes_{\beta,r} G)$–D imprimitivity bimodules, respectively, is proved in Lemma 3.2 (after again identifying $L(X) \rtimes_{u,r} G$ with $L(X \rtimes_{\gamma,r} G)$). Recall that $E_0 \coloneqq {}_{L(X)\rtimes_{u,r}G}\langle X_0 \oplus B_1, X_0 \oplus B_1 \rangle$ and that $E_0 \cdot (X_0 \oplus B_1)$ is dense in $(\overline{X_0 \oplus B_1})_D$. Since

$$\hat{p}\ {}_{L(X)\rtimes_{u,r}G}\left\langle \begin{pmatrix} x \\ b \end{pmatrix}, \begin{pmatrix} y \\ c \end{pmatrix} \right\rangle = {}_{L(X)\rtimes_{u,r}G}\left\langle \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ c \end{pmatrix} \right\rangle$$

implies

$$\hat{p}\ {}_{L(X)\rtimes_{u,r}G}\left\langle \begin{pmatrix} x \\ b \end{pmatrix}, \begin{pmatrix} y \\ c \end{pmatrix} \right\rangle \cdot \begin{pmatrix} z \\ d \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} \cdot \left\langle \begin{pmatrix} y \\ c \end{pmatrix}, \begin{pmatrix} z \\ d \end{pmatrix} \right\rangle_D \in \begin{pmatrix} X_0 \\ \{0\} \end{pmatrix},$$

we obtain that $\hat{p}\,(\overline{X_0 \oplus B_1})_D = (\overline{X_0 \oplus \{0\}})_D$. That $\hat{p}\,(\overline{X_0 \oplus B_1}) \cong {}_{A\rtimes_{\alpha,r}G}(\overline{X_0})_D$ is now clear because the inclusion of X_0 into $(\overline{X_0 \oplus \{0\}})_D$ preserves both inner products and the D-action. Similarly, $\hat{q}\,(\overline{X_0 \oplus B_1}) \cong {}_{B\rtimes_{\beta,r}G}(\overline{B_1})_D$.

Finally, to get the formula for the isomorphism, we need to chase through our identifications. Here, $f \cdot b$ means the left action of $f \in X \rtimes_{\gamma,r} G \subset L(X \rtimes_{\gamma,r} G) \cong L(X) \rtimes_{u,r} G$ on $b \in B_1 \subset X_0 \oplus B_1$. Thus we have a formula for the action provided

$$\begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} = {}_{L(X)\rtimes_{u,r}G}\left\langle \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ c \end{pmatrix} \right\rangle,$$

which means that f must have the form $f(s) = x \cdot \beta_s(c^*)\Delta(s)^{-1/2}$. If so,

$$\begin{aligned} \begin{pmatrix} f \cdot b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ b \end{pmatrix} &= \int_G \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}(s) \cdot \gamma_s \oplus \beta_s \begin{pmatrix} 0 \\ b \end{pmatrix} \Delta(s)^{1/2}\, ds \\ &= \int_G \begin{pmatrix} 0 & f(s) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \beta_s(b) \end{pmatrix} \Delta(s)^{1/2}\, ds \\ &= \int_G \begin{pmatrix} f(s) \cdot \beta_s(b)\Delta(s)^{1/2} \\ 0 \end{pmatrix} ds \\ &= \begin{pmatrix} \int_G f(s) \cdot \beta_s(b)\Delta(s)^{1/2}\, ds \\ 0 \end{pmatrix}, \end{aligned}$$

which gives the right formula. □

4. The symmetric imprimitivity theorem for graph algebras

A directed graph E consists of countable sets E^0 of vertices and E^1 of edges, and range and source maps $r, s : E^1 \to E^0$. A Cuntz–Krieger E-family in a C^*-algebra A consists of partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges and mutually orthogonal projections $\{p_v : v \in E^0\}$ such that

$$s_e^* s_e = p_{r(e)}, \quad s_e s_e^* \leqslant p_{s(e)} \quad \text{and} \quad p_v = \sum_{s(e)=v} s_e s_e^*$$
$$\text{whenever } 0 < |s^{-1}(v)| < \infty.$$

The graph C^*-algebra $C^*(E)$ is generated by a universal Cuntz–Krieger family $\{s_e, p_v\}$ (see [9] or [14], for example). We write E^* for the path space of E, and for $\mu \in E^*$ of length $|\mu|$ we write $s_\mu := s_{\mu_1} s_{\mu_2} \dots s_{\mu_{|\mu|}}$. The Cuntz–Krieger relations imply that every word in the s_e and s_f^* collapses to one of the form $s_\mu s_\nu^*$ for $\mu, \nu \in E^*$, and these are zero unless $r(\mu) = r(\nu)$. Thus,

$$X_0(E) := \operatorname{sp}\{s_\mu s_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu)\}$$

is a dense $*$-subalgebra of $C^*(E)$.

Suppose we have a left action of a (discrete) group G on E which is free on E^0 (and hence is free on E^1). The universal property of $C^*(E)$ implies that there is an induced action $\alpha : G \to \operatorname{Aut} C^*(E)$ such that $\alpha_g(s_e) = s_{g\cdot e}$ and $\alpha_g(p_v) = p_{g\cdot v}$. It is shown in [11, Section 1] that the action α is proper and saturated with respect to $X_0(E)$. Indeed, it is proved in [11, Lemma 1.1] that averaging over α gives a linear map $I_G : X_0(E) \to M(X_0(E))^\alpha$ whose range spans the generalized fixed-point algebra $C^*(E)^\alpha$, and that there is an isomorphism ϕ_G of the C^*-algebra $C^*(G\backslash E)$ of the quotient graph onto $C^*(E)^\alpha$; thus, it follows from Rieffel's theory that $C^*(E) \rtimes_{\alpha,r} G$ is Morita equivalent to $C^*(G\backslash E)$. The maps I_G and ϕ_G are also used in [11] to directly construct a bimodule implementing a symmetric imprimitivity theorem for the full crossed products, as follows.

Suppose we have commuting free actions of G and H on the left and right of E. Because the actions commute, they induce actions on the quotient graphs, and hence we have actions $\alpha : G \to \operatorname{Aut} C^*(E/H)$ and $\beta : H \to \operatorname{Aut} C^*(G\backslash E)$ on their C^*-algebras; it is safe to also use α and β for the actions on $X_0(E)$ and $C^*(E)$, because the maps ϕ_H and ϕ_G are then equivariant.

We write $k(H, X_0(G\backslash E))$ for the set of functions $f : H \to X_0(G\backslash E)$ with finite support. For $b \in k(H, X_0(G\backslash E))$, $c \in k(G, X_0(E/H))$ and $x, y \in X_0(E)$, we define

$$b \cdot x = \sum_{h \in H} \phi_G(b(h)) \beta_h(x), \tag{4.1}$$

$$x \cdot c = \sum_{g \in G} \alpha_g^{-1}(x \phi_H(c(g))), \tag{4.2}$$

$$_{k(H,X_0(G\backslash E))}\langle x,y\rangle(h)=\phi_G^{-1}\circ I_G(x\beta_h(y^*)),\tag{4.3}$$

$$\langle x,y\rangle_{k(G,X_0(E/H))}(g)=\phi_H^{-1}\circ I_H(x^*\alpha_g(y)).\tag{4.4}$$

Now [11, Theorem 2.1] states that $X_0(E)$ completes to give a Morita equivalence Z between $C^*(G\backslash E)\rtimes_\beta H$ and $C^*(E/H)\rtimes_\alpha G$.

We aim to apply Theorem 3.1 with $X=\overline{X_0(E)}$ the $C^*(G\backslash E)$–$(C^*(E)\rtimes_\alpha G)$ imprimitivity bimodule obtained by ignoring the action of H in (4.1)–(4.4). By [8, Corollary 3.3] or [11, Corollary 3.1], we have

$$C^*(E)\rtimes_\alpha G=C^*(E)\rtimes_{\alpha,r}G,$$

so we can view X as a module over the reduced crossed product.

Lemma 4.1. *The action β of H on $X_0(E)\subset C^*(E)$ induces actions β of H on $C^*(G\backslash E)$, β on X and $\beta\rtimes\mathrm{id}$ on $C^*(E)\rtimes_\alpha G$, and (X,H,β) is then a Morita equivalence between $(C^*(G\backslash E),H,\beta)$ and $(C^*(E)\rtimes_\alpha G,H,\beta\rtimes\mathrm{id})$. The action β on X is proper and saturated with respect to the pre-imprimitivity bimodule $_{X_0(G\backslash E)}X_0(E)_{k(G,X_0(E))}$.*

Proof. That β induces the actions on $C^*(G\backslash E)$ and $C^*(E)\rtimes_\alpha G$ is standard. Because β is compatible with the maps ϕ_G and I_G [11, Lemma 1.7], it is easy to check that β is compatible with the module actions and inner products. In particular, this implies that each β_h is isometric, and hence extends to an action on X implementing the desired Morita equivalence of systems. For the submodule $X_0(E)$, the functions in parts (1) and (2) of Definition 2.1 have finite support, and hence are trivially integrable. For $x,y\in X_0(E)$, the function $\langle x,y\rangle_D:G\to M(C^*(E))$ defined by

$$\langle x,y\rangle_D(g)=I_H(x^*\alpha_g(y))$$

also has finite support; the embedding of $M(C^*(E))\times_\alpha G$ in $M(C^*(E)\rtimes_\alpha G)$ carries this function into a multiplier $\langle x,y\rangle_D$ of $k(G,X_0(E))$ which satisfies Definition 2.1(3). Thus the action of H is proper. To see that it is saturated, we use [11, Lemma 1.4] to see that the function $\delta_h s_{G\cdot\mu}s_{G\cdot\nu}^*$ in $k(H,X_0(G\backslash E))$ is given by

$$\delta_h s_{G\cdot\mu}s_{G\cdot\nu}^*=\delta_h\phi_G^{-1}\circ I_G(s_\mu s_\nu^*)=\langle x,y\rangle_D,$$

when $x=s_\mu s_\nu^*$ and $y=p_{s(\nu)\cdot h}$. □

Applying Theorem 3.1 to (X,H,β) gives a $(C^*(E)\rtimes_{\alpha\times\beta}(G\times H))$–$D$ imprimitivity bimodule $\overline{B_1}$, a $(C^*(G\backslash E)\rtimes_\beta H)$–$D$ imprimitivity bimodule $\overline{X_0}$, and a decomposition isomorphism. The space $X_0(E)$ underlies both $\overline{X_0}$ and the bimodule Z of [11]. Here X_0 is really a bimodule over $k(H,X_0(G\backslash E))$ and the generalized fixed-point algebra $D\subset M(C^*(E)\rtimes_\alpha G)$; when we use $\phi_H\rtimes\mathrm{id}$ to identify D with $C^*(E/H)\rtimes_\alpha G$, our formulas convert to the ones (4.1)–(4.4) used in [11]. Thus:

Theorem 4.2. *The bimodule* $_{k(H,X_0(G\backslash E))}X_0(E)_{k(G,X_0(E/H))}$ *described in* (4.1)–(4.4) *completes to give an imprimitivity bimodule which implements a Morita equivalence between* $C^*(G\backslash E)\rtimes_{\beta,r}H$ *and* $C^*(E/H)\rtimes_{\alpha,r}G$.

Comparing this bimodule to the one for the full crossed products allows us to settle a question left open in [11, Remark 3.2].

Corollary 4.3. *Suppose that G and H act freely on the left and right of a directed graph E, and let* α *and* β *denote the induced actions on* $C^*(E/H)$ *and* $C^*(G\backslash E)$. *Then regular representations of* $(C^*(E/H), G, \alpha)$ *are faithful if and only if regular representations of* $(C^*(G\backslash E), H, \beta)$ *is faithful.*

Proof. Let I be the kernel of the quotient map from $C^*(E/H)\rtimes_\alpha G$ to $C^*(E/H)\rtimes_{\alpha,r}G$. Then, by the Rieffel correspondence [15, Section 3.3], there are a closed submodule W of the bimodule Z of [11] and an ideal $J = Z\text{-Ind}\, I$ in $C^*(G\backslash E)\rtimes_\beta H$ such that Z/W is a $(C^*(G\backslash E)\rtimes_\beta H)/J$–$(C^*(E/H)\rtimes_\alpha G)/I$ imprimitivity bimodule. In particular, this implies that the semi-norms on $X_0(E)\subset Z$ induced by the quotient norms on $(C^*(G\backslash E)\rtimes_\beta H)/J$ and $(C^*(E/H)\rtimes_\alpha G)/I$ coincide [15, Proposition 3.11]. The semi-norm coming from the right inner product is that induced by the reduced norm on $k(G, X_0(E/H))$. However, we know by applying [15, Proposition 3.11] to the bimodule of Theorem 4.2 that this coincides with the semi-norm induced by the left inner product and the reduced norm on $k(H, X_0(G\backslash E))$. Thus, the semi-norm on $k(H, X_0(G\backslash E))$ pulled back from the quotient $(C^*(G\backslash E)\rtimes_\beta H)/J$ is the reduced semi-norm, the quotient is the reduced crossed product, and J is the kernel of the quotient map onto $C^*(G\backslash E)\rtimes_{\beta,r}H)$. Since $I = 0$ if and only if $Z\text{-Ind}\, I = 0$, the result follows. $\square$

To complete the analysis, we identify $\overline{B_1}$ and check that the decomposition of Theorem 3.1 gives an isomorphism between the tensor-product equivalence of [11, Theorem 1.9] and that of Theorem 4.2.

The algebra B_1 is spanned by the range of the inner product on $X_0(E)$, and since

$$\delta_g s_\mu s_\nu^* = \langle s_\nu s_\mu^*, p_{g^{-1}\cdot s(\nu)} \rangle_{k(G,X_0(E))},$$

this is all of $k(G, X_0(E))$. This is also a dense subspace of the Combes bimodule $Y\rtimes_\alpha G$, where Y is the $(C^*(E)\rtimes_\beta H)$–$C^*(E/H)$ imprimitivity bimodule based on $Y_0 := X_0(E)$ obtained by ignoring G in (4.1)–(4.4). To see that $\overline{B_1}$ is isomorphic to $Y\rtimes_{\alpha,r}G$, we need to note that the map $\phi_H\rtimes\text{id}$ is an isomorphism of $C^*(E/H)\rtimes_{\alpha,r}G$ onto the fixed-point algebra D, and check that the inner products and module actions match up.

Let $a, b, c\in k(G, X_0(E))$. To verify that the D-valued inner product on B_1 and the $C^*(E/H)\rtimes_{\alpha,r}G$-valued inner product on $Y\rtimes_{\alpha,r}G$ agree modulo the isomorphism

$\phi_H \rtimes \mathrm{id}$, note that

$$a \cdot \langle b\,, c\rangle_D = \sum_{h\in H} a(\beta \rtimes \mathrm{id})_h(b^* c), \tag{4.5}$$

where the two products on the right are convolution in $C^*(E) \rtimes_\alpha G$. Thus $a \cdot \langle b, c\rangle_D(t)$ can be written as a triple sum over $H \times G \times G$. On the other hand, using (3.6) and (3.8), we get

$$a \cdot \langle b, c\rangle_{C^*(E/H)\rtimes_{\alpha,r} G}(t) = \sum_{s\in G}\sum_{l\in G} a(s)\alpha_{s^{-1}l}(\langle b(l), c(ls^{-1}t)\rangle_{C^*(E/H)}),$$

which is the same as (4.5) after using ϕ_H to identify $C^*(E/H)$ and $C^*(E)^\beta$ and applying (4.4).

To see that the left inner products coincide, we start with the inner product from $Y \rtimes_{\alpha,r} G$: from (3.7) we have

$$_{(C^*(E)\rtimes_\beta H)\rtimes_{\alpha,r} G}\langle b, c\rangle(t, h) = \sum_{s\in G} {}_{C^*(E)\rtimes_\beta H}\langle b(s), \alpha_t(c(t^{-1}s))\rangle(h),$$

which, using (4.3) and the isomorphism ϕ_H, is $\sum_{s\in G} b(s)\beta_h(\alpha_t(c(t^{-1}s)^*))$. But we can recognize this as the convolution product $b\beta_h(c^*)$ evaluated at t, which is $_{(C^*(E)\rtimes_\alpha G)\rtimes_{\beta,r} H}\langle b, c\rangle(h, t)$.

Using (3.6) it is straightforward to check that the right action of $C^*(E/H) \rtimes_{\alpha,r} G$ on $Y \rtimes_{\alpha,r} G$ is the same as the action of D as multipliers on B_1. If $f \in k(G \times H, X_0(E))$ then using (3.5) we get

$$f \cdot a(t) = \sum_{s\in G} f(s, \cdot) \cdot \alpha_s(a(s^{-1}t)),$$

which, using (4.1) and ϕ_G, reduces to the left action of f on B_1 given by Lemma 2.17. Thus $\overline{B_1}$ and $Y \rtimes_{\alpha,r} G$ are isomorphic.

The decomposition isomorphism of $(X \rtimes_{\beta,r} H) \otimes_{C^*(E)\rtimes_r (H\times G)} (Y \rtimes_{\alpha,r} G)$ onto the reduced symmetric imprimitivity bimodule of Theorem 4.2 is given by

$$\Omega(e \otimes b) = \sum_{h\in H} e(h) \cdot (\beta \rtimes \mathrm{id})_h(b),$$

where the action is that of $E_0 \subset k(G, X_0(E/H))$ on X_0. Working out the formulas in terms of the product in $X_0(E) \subset C^*(E)$ gives

$$\Omega(e \otimes b) = \sum_{h\in H}\sum_{g\in G} \alpha_g^{-1}(e(h)\phi_H(\beta_h(b(g)))), \tag{4.6}$$

and the functions e, b of the required form span $k(H, X_0)$ and $k(G, Y_0)$, respectively. Hence:

Corollary 4.4. *The map* $\Omega : k(H, X_0) \otimes k(G, Y_0) \to X_0$ *defined by* (4.6) *extends to an isomorphism of* $(X \rtimes_{\beta,r} H) \otimes_{C^*(E) \rtimes_r (H \times G)} (Y \rtimes_{\alpha,r} G)$ *onto the reduced symmetric imprimitivity bimodule of Theorem* 4.2.

Remark 4.5. A similar analysis can be carried out for the symmetric imprimitivity theorem of [13], and yields an isomorphism of the form (1.1) for the bimodule Z which implements the reduced version of the symmetric imprimitivity theorem, as in [12] or [6, Corollary 2]. With a bit of work, one can check that this isomorphism is given on suitable dense subspaces by the same formula as that of [7, Lemma 4.8].

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