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Induction in stages for crossed products of C^*-algebras by maximal coactions ☆

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Abstract

Let δ be a maximal coaction of a locally compact group G on a C^*-algebra B, and let N and H be closed normal subgroups of G with $N \subseteq H$. We show that the process $\mathrm{Ind}_{G/H}^{G}$ which uses Mansfield's bimodule to induce representations of $B \rtimes_\delta G$ from those of $B \rtimes_{\delta|} (G/H)$ is equivalent to the two-stage induction process $\mathrm{Ind}_{G/N}^{G} \circ \mathrm{Ind}_{G/H}^{G/N}$. The proof involves a calculus of symmetric imprimitivity bimodules which relates the bimodule tensor product to the fibred product of the underlying spaces.

Keywords: C^*-algebras; Crossed products; Induced representations; Maximal coactions

1. Introduction

Induction is a method of constructing representations which is important in many different situations. The modern C^*-algebraic theory of induction has its roots in Mackey's work on the induced representations of locally compact groups, which culminated in the Mackey machine

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for computing the irreducible unitary representations of a locally compact group [15], and in Rieffel's recasting of the Mackey machine in the language of Morita equivalence—indeed, Rieffel developed his concept of Morita equivalence for C^*-algebras specifically for this purpose [19,20]. Takesaki adapted Mackey's construction to the context of dynamical systems (A, G, α) in which a locally compact group G acts by automorphisms of a C^*-algebra A [21], and the full strength of the modern theory was achieved when Green applied Rieffel's ideas to dynamical systems [7]. Takesaki and Green showed in particular how to induce a covariant representation (π, U) of the system $(A, H, \alpha|)$ associated to a closed subgroup H of G to a covariant representation $\mathrm{Ind}_H^G(\pi, U)$ of (A, G, α).

These various theories of induced representations share the following fundamental properties:

Imprimitivity: There is an imprimitivity theorem which characterises the representations which are unitarily equivalent to induced representations.
Regularity: The representations induced from the trivial subgroup $\{e\}$ are precisely the regular representations, up to unitary equivalence.
Induction in stages: If K and H are closed subgroups of G with $K \subseteq H$, then $\mathrm{Ind}_H^G \circ \mathrm{Ind}_K^H = \mathrm{Ind}_K^G$, up to unitary equivalence.

Green's formulation of induced representations uses the bijection $(\tau, V) \mapsto \tau \rtimes V$ between covariant representations of (A, G, α) and representations of the crossed product C^*-algebra $A \rtimes_\alpha G$, and his induction process is implemented by (what we now call) a right-Hilbert $(A \rtimes_\alpha G)$–$(A \rtimes_{\alpha|} H)$ bimodule $X_H^G(\alpha)$: if (π, U) is a covariant representation of $(A, H, \alpha|)$ on a Hilbert space $\mathcal{H}$, then the induced representation $\mathrm{Ind}_G^H(\pi \times U)$ of $A \rtimes_\alpha G$ acts in $X_G^H(\alpha) \otimes_{A \rtimes_{\alpha|} H} \mathcal{H}$ through the left action of $A \rtimes_\alpha G$ on $X_H^G(\alpha)$. Green proved that one can fatten up the left action of $A \rtimes_\alpha G$ to an action of $(A \otimes C_0(G/H)) \rtimes_{\alpha \otimes \mathrm{lt}} G$; with this new left action, the bimodule becomes a Morita equivalence. The resulting imprimitivity theorem says that a representation (τ, V) of (A, G, α) on some Hilbert space $\mathcal{H}_0$ is induced from a representation of $(A, H, \alpha|)$ if and only if there is a representation μ of $C_0(G/H)$ on $\mathcal{H}_0$ which commutes with $\tau(A)$ and gives a covariant representation (μ, V) for the action lt of G by left translation on $C_0(G/H)$ [7, Theorem 6]. The general theory of Hilbert bimodules guarantees that the induction process has good functorial properties, and Green proved induction-in-stages by constructing a bimodule isomorphism of $X_H^G(\alpha) \otimes_{A \rtimes_{\alpha|} H} X_K^H(\alpha|)$ onto $X_K^G(\alpha)$ [7, Proposition 8].

In nonabelian duality, one works with coactions of locally compact groups on C^*-algebras: the motivating example is the dual coaction $\hat{\alpha}$ of G on a crossed product $A \rtimes_\alpha G$, from which one can recover a system Morita equivalent to (A, G, α) by taking a second crossed product $(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} G$. The crossed product $B \rtimes_\delta G$ of a C^*-algebra B by a coaction δ of G on B is universal for a class of covariant representations (π, μ) consisting of compatible representations of B and $C_0(G)$ on the same Hilbert space. Induced representations of crossed products by coactions were first constructed by Mansfield [16], who associated to each closed normal amenable subgroup N a right-Hilbert $(B \rtimes_\delta G)$–$(B \rtimes_{\delta|} (G/N))$ bimodule, and thereby plugged into Rieffel's general framework. Mansfield checked that inducing from $B \rtimes_{\delta|} (G/G) = B$ gave the generally accepted class of regular representations [16, Proposition 21], and proved an elegant imprimitivity theorem: a representation τ of $B \rtimes_\delta G$ is induced from a representation of $B \rtimes_{\delta|} (G/N)$ if and only if there is a unitary representation V of N such that (τ, V) is covariant for the dual action $\hat{\delta}|$ of N. Induction-in-stages was later proved in [14, Corollary 4.2].

The hypothesis of amenability appears in Mansfield's theory because his construction is intrinsically spatial, and the Morita equivalence underlying his imprimitivity theorem involves the reduced crossed product $(B \rtimes_\delta G) \rtimes_{\hat{\delta},r} N$. Subsequent authors have shown how to lift the amenability and normality hypotheses [8,12], but the resulting imprimitivity theorems still use the reduced crossed product by the dual action, and are therefore not well suited to applications involving covariant representations. In an effort to produce a theory which is more friendly to full crossed products by actions, Echterhoff, Kaliszewski and Quigg have proposed the study of *maximal* coactions [5], which include the dual coactions and certain other coactions constructed from them [13, §7].

Kaliszewski and Quigg have recently shown that for a maximal coaction δ of G on a C^*-algebra B and any closed normal subgroup N of G, the crossed product $B \rtimes_{\delta|} (G/N)$ by the restriction of δ is Morita equivalent, via a *Mansfield bimodule* we will denote by $Y^G_{G/N}(\delta)$, to the full crossed product $(B \rtimes_\delta G) \rtimes_{\hat{\delta}|} N$. Dropping the left action of N on their Morita equivalence gives a right-Hilbert $(B \rtimes_\delta G)$–$(B \rtimes_{\delta|} (G/N))$ bimodule which can be used to define induced representations $\operatorname{Ind}^G_{G/N}(\pi \rtimes \mu)$, and Theorem 5.3 of [13] gives an imprimitivity theorem for this induction process. Our goal in this paper is to prove regularity and induction-in-stages for this induction process of Kaliszewski and Quigg. Regularity is straightforward, and is addressed in the short Section 2. Proving induction-in-stages—the assertion that $\operatorname{Ind}^G_{G/H}$ is equivalent to $\operatorname{Ind}^G_{G/N} \circ \operatorname{Ind}^{G/N}_{G/H}$—occupies most of the rest of the paper. Specifically, we will prove:

Theorem 1.1. *Let $\delta : B \to M(B \otimes C^*(G))$ be a maximal coaction of a locally compact group G on a C^*-algebra B. Also let N and H be closed normal subgroups of G with $N \subseteq H$. Then the following diagram of right-Hilbert bimodules commutes*:

$$\begin{array}{ccc} B \rtimes_\delta G & \xrightarrow{\quad Y^G_{G/H}(\delta) \quad} & B \rtimes_{\delta|} (G/H). \\ & {\scriptstyle Y^G_{G/N}(\delta)} \searrow \qquad \nearrow {\scriptstyle Y^{G/N}_{G/H}(\delta|)} & \\ & B \rtimes_{\delta|} (G/N) & \end{array} \tag{1.1}$$

Equivalently,

$$Y^G_{G/H}(\delta) \cong Y^G_{G/N}(\delta) \otimes_{B \rtimes_{\delta|}(G/N)} Y^{G/N}_{G/H}(\delta|)$$

as right-Hilbert $(B \rtimes_\delta G)$–$(B \rtimes_{\delta|} (G/H))$ bimodules.

Here, both $Y^G_{G/N}(\delta)$ and $Y^G_{G/H}(\delta)$ are Mansfield bimodules defined using the coaction δ of G on B. The bimodule $Y^{G/N}_{G/H}(\delta|)$ is defined using the restricted coaction $\delta|_{G/N}$ of G/N on B and the normal subgroup $H/N \subseteq G/N$, and we have identified the quotient $(G/N)/(H/N)$ with G/H.

The Mansfield bimodule is defined in [13] as a tensor product of three other bimodules (see Remark 6.3); thus proving that (1.1) commutes using first principles would involve gluing a different commutative square onto each of the arrows in (1.1), and then proving that the resulting outer figure—which would involve a terrifying nine bimodules—commutes. More importantly, this approach obscures the fundamental idea behind the definition of the Y's, which is to pass to the second-dual coaction (maximal coactions are precisely those for which full crossed-product duality holds) and then invoke the symmetric imprimitivity theorem of [17].

Thus, our general strategy for the proof of Theorem 1.1 will appeal to this underlying idea rather than the definition itself. We will use the naturality of the Mansfield bimodules (see [6] for the technical meaning of this) to reduce to the case where δ is a dual coaction. If $\delta = \hat{\alpha}$ is a dual coaction, it is known [13, Proposition 6.5] that the Mansfield bimodules $Y^{G}_{G/N}(\hat{\alpha})$ and $Y^{G}_{G/H}(\hat{\alpha})$ appearing in (1.1) can be replaced by bimodules $Z^{G}_{G/N}(\alpha)$ and $Z^{G}_{G/H}(\alpha)$ constructed using the symmetric imprimitivity theorem. In Theorem 4.1, we extend this result by showing that $Y^{G/N}_{G/H}(\hat{\alpha}|)$ is isomorphic to the symmetric imprimitivity bimodule constructed in [11, Proposition 3.3], which we denote by $Z^{G/N}_{G/H}(\alpha)$. Combining various results from the literature gives an analog of (1.1) for the Z's; and then we can assemble all of our intermediate results in Section 6 to complete the proof of Theorem 1.1.

Because the restriction $\hat{\alpha}|_{G/N}$ need not be the dual of an action of G/N, the isomorphism $Y^{G/N}_{G/H}(\hat{\alpha}|) \cong Z^{G/N}_{G/H}(\alpha)$ is not simply another application of [13, Proposition 6.5]; indeed, establishing this result occupies most of the present paper. Rather than dealing directly with the definition of the Mansfield bimodule, we appeal to [13, Corollary 6.4], which shows that $Y^{G/N}_{G/H}(\hat{\alpha}|)$ can be "factored" into a tensor product involving Green and Katayama imprimitivity bimodules. The desired isomorphism follows when we show (Theorem 4.2) that $Z^{G/N}_{G/H}(\alpha)$ can be factored the same way. The preparation for the proof of Theorem 4.2 involves identifying each of the three imprimitivity bimodules in question with a bimodule constructed from the symmetric imprimitivity theorem; this is carried out in Section 4. The proof itself occupies Section 5, where we apply a calculus, developed in Section 3, which allows the tensor product of such bimodules to be studied at the level of the spaces from which they were constructed.

We expect that this calculus will be of independent interest in the future. To further illustrate its utility, in Section 7 we apply it to the balanced tensor product of two one-sided versions of the symmetric imprimitivity, thus recovering the isomorphism of the tensor product and the symmetric version from [9, Lemma 4.8] on the level of spaces.

1.1. Notation and conventions

Our reference for the theory of crossed products by actions and coactions is [6]. We follow the conventions of [13] for coactions; in particular, all our coactions are nondegenerate and maximal.

We write λ and ρ for the left and right regular representations, respectively, of a group G on $L^2(G)$. If N is a normal subgroup of G we write $\lambda^{G/N}$ for the quasi-regular representation of G on $L^2(G/N)$ and M or $M^{G/N}$ for the representation of $C_0(G/N)$ on $L^2(G/N)$ by multiplication operators, so that $(\lambda^{G/N}_r\xi)(sN) = \xi(r^{-1}sN)$ and $M(f)\xi(sN) = f(sN)\xi(sN)$ for $\xi \in L^2(G/N)$, $f \in C_0(G/N)$ and $r, s \in G$.

Let $\alpha : G \to \operatorname{Aut} A$ be a continuous action of G by automorphisms of a C^*-algebra A, and write lt and rt for the actions of G on $C_0(G)$ by left and right translation, so that

$$\mathrm{lt}_s(f)(t) = f\bigl(s^{-1}t\bigr) \quad \text{and} \quad \mathrm{rt}_s(f)(t) = f(ts) \quad \text{for } f \in C_0(G) \text{ and } s, t \in G.$$

If N is a closed normal subgroup of G, then there is a natural isomorphism of $(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G$ onto $(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}|} G/N$ ([4, Lemma 2.3]; see also [6, Proposition A.63 and Theorem A.64]). Representations of both C^*-algebras come from suitably covariant representations π, μ, and U of A, $C_0(G/N)$, and G (respectively) on the same Hilbert space; the isomorphism carries $(\pi \otimes \mu) \rtimes U$ to $(\pi \rtimes U) \rtimes \mu$ and for this reason we refer to it (and related maps) as the *canonical isomorphism*.

If A and B are C^*-algebras, a right-Hilbert A–B bimodule is a right-Hilbert B-module X together with a homomorphism φ of A into the C^*-algebra $\mathcal{L}(X)$ of adjointable operators on X; in practice, we suppress φ and write $a \cdot x$ for $\varphi(a)x$. As in [6], we view a right-Hilbert A–B bimodule X as a morphism from A to B, and say that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad X \quad} & B \\ {\scriptstyle Z}\big\downarrow & & \big\downarrow{\scriptstyle Y} \\ C & \xrightarrow[\quad W \quad]{} & D \end{array}$$

commutes if $X \otimes_B Y$ and $Z \otimes_C W$ are isomorphic as right-Hilbert A–D bimodules. If $\varphi : A \to C$ and $\psi : B \to D$ are isomorphisms, then the right-Hilbert C–D bimodule X' obtained from X by adjusting the coefficient algebras using φ and ψ is by definition the bimodule such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad X \quad} & B \\ {\scriptstyle \varphi}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \psi} \\ C & \xrightarrow[\quad X' \quad]{} & D \end{array}$$

commutes. (Formally, the left vertical arrow, for example, is the A–C bimodule A with $a \cdot b = ab$, $\langle a, b\rangle_C = \varphi(a^*b)$ and $a \cdot c = a\varphi^{-1}(c)$ for $a, b \in A$ and $c \in C$.) If B is contained in the multiplier algebra $M(A)$ of A, we denote by Res the right-Hilbert B–A bimodule A, where

$$b \cdot a = ba, \quad a \cdot c = ac, \quad \langle a, c\rangle_A = a^*c \quad \text{and} \quad {}_B\langle a, c\rangle d = ac^*d$$

for $a, c, d \in A$ and $b \in B$.

We will often write ${}_*\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_*$ for the left- and right-inner products, respectively, in an imprimitivity bimodule, and trust that it is clear from context in which algebra the values lie.

2. Regularity

In the coaction context, *regularity* means that the regular representations are, up to unitary equivalence, precisely those induced from the trivial quotient group G/G:

Proposition 2.1. *Let $\delta : B \to M(B \otimes C^*(G))$ be a maximal coaction of a locally compact group G on a C^*-algebra B. Then for each nondegenerate representation π of B on a Hilbert space $\mathcal{H}$, the representation $\operatorname{Ind}_{G/G}^G(\pi)$ of $B \rtimes_\delta G$ induced using the Mansfield bimodule $Y_{G/G}^G(\delta)$ is unitarily equivalent to the regular representation $((\pi \otimes \lambda) \circ \delta) \rtimes (1 \otimes M)$ of $B \rtimes_\delta G$ on $\mathcal{H} \otimes L^2(G)$.*

Since δ is a maximal coaction of G on B, by definition of maximality [5, Definition 3.1], the canonical surjection

$$(\mathrm{id} \otimes \lambda) \circ \delta \rtimes (1 \otimes M) \rtimes (1 \otimes \rho) : B \rtimes_\delta G \rtimes_{\hat{\delta}} G \to B \otimes \mathcal{K}\big(L^2(G)\big) \tag{2.1}$$

is an isomorphism. This makes the $B \otimes \mathcal{K}(L^2(G))$–$B$ imprimitivity bimodule $B \otimes L^2(G)$ into a $(B \rtimes_\delta G \rtimes_{\hat\delta} G)$–$B$ imprimitivity bimodule which we call the *Katayama bimodule* [13, Definition 4.1], and which we denote by $K(\delta)$. By [13, Corollary 6.2], the imprimitivity bimodules $K(\delta)$ and $Y^G_{G/G}(\delta)$ are isomorphic, so to prove the proposition it suffices to deal with $K(\delta)$.

Proof. It is straightforward to see that the map θ determined by

$$\theta(b \otimes \xi \otimes h) = \pi(b)h \otimes \xi, \quad \text{where } b \in B,\ h \in \mathcal{H},\ \xi \in L^2(G),$$

extends to a unitary isomorphism of $K(\delta) \otimes_B \mathcal{H}$ onto $\mathcal{H} \otimes L^2(G)$.

Denote by j_B and $j_{C(G)}$ the canonical maps of B and $C_0(G)$ into $M(B \rtimes_\delta G)$. To see that θ intertwines the induced representation and the regular representation, it suffices to check that:

(1) $\theta(\mathrm{Ind}^G_{G/G}(\pi)(j_B(b))\zeta) = (\pi \otimes \lambda(\delta(b)))\theta(\zeta)$, and
(2) $\theta(\mathrm{Ind}^G_{G/G}(\pi)(j_{C(G)}(f))\zeta) = (1 \otimes M(f))\theta(\zeta)$

for $b \in B$, $f \in C_0(G)$, and $\zeta \in K(\delta) \otimes_B \mathcal{H}$; it further suffices to consider ζ of the form $a \otimes \eta \otimes h$ for $a \in B$, $\eta \in L^2(G)$ and $h \in \mathcal{H}$. Verifying (2) is straightforward. To check (1), we use nondegeneracy to write $\eta = \lambda(c)\xi$ for $c \in C^*(G)$ and $\xi \in L^2(G)$; then $\delta(b)(1 \otimes c) \in B \otimes C^*(G)$, and we can approximate it by a sum $\sum_{j=1}^n b_j \otimes c_j \in B \otimes C^*(G)$. Now we can do an approximate calculation:

$$\begin{aligned}
\theta\big(\mathrm{Ind}^G_{G/G}(\pi)\big(i_B(b)\big)\big(a \otimes \lambda(c)\xi \otimes h\big)\big) &= \theta\big(\big(\mathrm{id} \otimes \lambda\big(\delta(b)\big)\big(a \otimes \lambda(c)\xi\big)\big) \otimes h\big) \\
&= \theta\big(\big(\mathrm{id} \otimes \lambda\big(\delta(b)(1 \otimes c)\big)(a \otimes \xi)\big) \otimes h\big) \\
&\sim \sum_{j=1}^n \theta\big(\big(\mathrm{id} \otimes \lambda(b_j \otimes c_j)(a \otimes \xi)\big) \otimes h\big) \\
&= \sum_{j=1}^n \pi(b_j a)h \otimes \lambda(c_j)\xi \\
&= \pi \otimes \lambda\left(\sum_{j=1}^n b_j \otimes c_j\right)\big(\pi(a)h \otimes \xi\big) \\
&\sim \pi \otimes \lambda\big(\delta(b)(1 \otimes c)\big)\big(\pi(a)h \otimes \xi\big) \\
&= \pi \otimes \lambda\big(\delta(b)\big)\big(\pi(a)h \otimes \lambda(c)\xi\big) \\
&= \pi \otimes \lambda\big(\delta(b)\big)\theta\big(a \otimes \lambda(c)\xi \otimes h\big);
\end{aligned}$$

since the approximations can be made arbitrarily accurate, this implies (1). □

3. A calculus for symmetric-imprimitivity bimodules

The set-up for the symmetric imprimitivity theorem of [17] is that of commuting free and proper actions of locally compact groups K and L on the left and right, respectively, of a locally compact space P. In addition, there are commuting actions σ and η of K and L on a

C^*-algebra A. We sum up this set-up by saying that $({}_K P_L, A, \sigma, \eta)$ is *symmetric imprimitivity data*, and we represent this schematically with the diagram

$$\begin{array}{ccccc} & & P & & \\ & \nearrow & & \nwarrow & \\ K & & & & L \\ & \sigma\searrow & & \swarrow\eta & \\ & & A & & \end{array}$$

The *induced algebra* $\operatorname{Ind}_L^P \eta$ consists of all functions $f \in C_{\mathrm{b}}(P, A)$ such that

$$f(p \cdot t) = \eta_t^{-1}\big(f(p)\big) \quad \text{for } t \in L \text{ and } p \in P \quad \text{and} \quad \big(pL \mapsto \|f(p)\|\big) \in C_0(P/L).$$

Similarly, $\operatorname{Ind}_K^P \sigma$ consists of all functions $f \in C_{\mathrm{b}}(P, A)$ such that

$$f(s \cdot p) = \sigma_s\big(f(p)\big) \quad \text{for } s \in K \text{ and } p \in P \quad \text{and} \quad \big(Kp \mapsto \|f(p)\|\big) \in C_0(K \setminus P).$$

$\operatorname{Ind}_L^P \eta$ admits the diagonal action $\sigma \otimes \mathrm{lt}$ of K, and $\operatorname{Ind}_K^P \sigma$ admits the diagonal action $\eta \otimes \mathrm{rt}$ of L. The symmetric imprimitivity theorem [17, Theorem 1.1] says that $C_{\mathrm{c}}(P, A)$ can be completed to a

$$\big(\operatorname{Ind}_L^P \eta \rtimes_{\sigma\otimes\mathrm{lt}} K\big)\text{–}\big(\operatorname{Ind}_K^P \sigma \rtimes_{\eta\otimes\mathrm{rt}} L\big)$$

imprimitivity bimodule. We denote this bimodule by $W({}_K P_L, A, \sigma, \eta)$, or more compactly, by $W(P)$.

In this section we consider two sets of symmetric imprimitivity data, $({}_K P_L, A, \sigma, \eta)$ and $({}_L Q_G, A, \xi, \tau)$, which are compatible in a way that ensures there is an isomorphism Φ of $\operatorname{Ind}_K^P \sigma \rtimes_{\eta\otimes\mathrm{rt}} L$ onto $\operatorname{Ind}_G^Q \tau \rtimes_{\xi\otimes\mathrm{lt}} L$. Thus we can form the imprimitivity bimodule $W(P) \otimes_\Phi W(Q)$, which is by definition the imprimitivity bimodule such that the diagram

$$\begin{array}{ccc} \operatorname{Ind}_L^P \eta \rtimes_{\sigma\otimes\mathrm{lt}} K & \xrightarrow{W(P)\otimes_\Phi W(Q)} & \operatorname{Ind}_L^Q \zeta \rtimes_{\tau\otimes\mathrm{rt}} G \\ {\scriptstyle W(P)}\big\downarrow & & \big\uparrow{\scriptstyle W(Q)} \\ \operatorname{Ind}_K^P \sigma \rtimes_{\eta\otimes\mathrm{rt}} L & \underset{\cong}{\xrightarrow{\Phi}} & \operatorname{Ind}_G^Q \tau \rtimes_{\zeta\otimes\mathrm{lt}} L \end{array}$$

of imprimitivity bimodules commutes. Theorem 3.1 will show that $W(P) \otimes_\Phi W(Q)$ can be replaced with an imprimitivity bimodule based on a single set of symmetric imprimitivity data, thus giving an easy way of calculating, at the level of spaces, the isomorphism class of the balanced tensor product.

Suppose $\varphi : K \setminus P \to Q/G$ is a homeomorphism which is L-equivariant in the sense that

$$\varphi(K \cdot p \cdot t) = t^{-1} \cdot \varphi(K \cdot p) \quad \text{for all } t \in L \text{ and } p \in P, \tag{3.1}$$

and let $P \times_\varphi Q := \{(p,q) \in P \times Q\colon \varphi(K \cdot p) = q \cdot G\}$ be the fibred product. We define

$$P \,\#_\varphi\, Q := (P \times_\varphi Q)/L, \tag{3.2}$$

where the action of L on $P \times_\varphi Q$ is via the diagonal action $(p,q)\cdot t := (p \cdot t, t^{-1} \cdot q)$. We will use $[p,q]$ to denote the class of (p,q) in $P \,\#_\varphi\, Q$; we will write $P \# Q$ for $P \,\#_\varphi\, Q$ when there is no risk of confusion.

Theorem 3.1. *Suppose K, L and G are locally compact groups, and suppose that $({}_K P_L, A, \sigma, \eta)$ and $({}_L Q_G, A, \zeta, \tau)$ are symmetric imprimitivity data. In addition, suppose there is an L-equivariant homeomorphism $\varphi : K \backslash P \to Q/G$ as at* (3.1)*, and that there are continuous maps*

$$\widetilde{\sigma} : P \to \operatorname{Aut} A \quad \textit{and} \quad \widetilde{\tau} : Q \to \operatorname{Aut} A$$

such that, for $p \in P$, $q \in Q$, $k \in K$, $m \in G$ and $t \in L$,

$$\widetilde{\sigma}_{k\cdot p\cdot t} = \sigma_k \widetilde{\sigma}_p \zeta_t, \tag{3.3}$$

$$\widetilde{\tau}_{t\cdot q\cdot m} = \eta_t \widetilde{\tau}_q \tau_m, \quad \textit{and} \tag{3.4}$$

$$\zeta,\ \sigma \textit{ and } \widetilde{\sigma} \textit{ commute with } \eta,\ \widetilde{\tau} \textit{ and } \tau. \tag{3.5}$$

Then $P \,\#_\varphi\, Q$, as defined at (3.2)*, admits commuting free and proper actions of K and G, and there are isomorphisms*

$$\Phi : \operatorname{Ind}_K^P \sigma \rtimes_{\eta\otimes \mathrm{rt}} L \to \operatorname{Ind}_G^Q \tau \rtimes_{\zeta\otimes \mathrm{lt}} L, \tag{3.6}$$

$$\Phi_\sigma : \operatorname{Ind}_K^{P\#Q} \sigma \rtimes_{\tau\otimes \mathrm{rt}} G \to \operatorname{Ind}_L^Q \zeta \rtimes_{\tau\otimes \mathrm{rt}} G, \quad \textit{and} \tag{3.7}$$

$$\Phi_\tau : \operatorname{Ind}_G^{P\#Q} \tau \rtimes_{\sigma\otimes \mathrm{lt}} K \to \operatorname{Ind}_L^P \eta \rtimes_{\sigma\otimes \mathrm{lt}} K, \tag{3.8}$$

such that the diagram

$$\begin{array}{ccc}
\operatorname{Ind}_G^{P\#Q} \tau \rtimes_{\sigma\otimes \mathrm{lt}} K & \xrightarrow{\;W(P\#Q)\;} & \operatorname{Ind}_K^{P\#Q} \sigma \rtimes_{\tau\otimes \mathrm{rt}} G \\
{\scriptstyle \Phi_\tau}\downarrow \cong & & \cong \downarrow {\scriptstyle \Phi_\sigma} \\
\operatorname{Ind}_L^P \eta \rtimes_{\sigma\otimes \mathrm{lt}} K & \xrightarrow{\;W(P)\otimes_\Phi W(Q)\;} & \operatorname{Ind}_L^Q \zeta \rtimes_{\tau\otimes \mathrm{rt}} G
\end{array} \tag{3.9}$$

of imprimitivity bimodules commutes.

In the proof of Theorem 3.1 we will use the following lemma to establish the isomorphisms (3.6)–(3.8). In part (2) of the lemma, φ_* denotes the natural isomorphism of $C_0(Q/G, A)$ onto $C_0(K \setminus P, A)$ induced by φ.

Lemma 3.2. *Assume the hypotheses of Theorem* 3.1.

(1) *There are isomorphisms* $\varphi_\sigma : \operatorname{Ind}_K^{P\#Q} \sigma \to \operatorname{Ind}_L^Q \zeta$ *and* $\varphi_\tau : \operatorname{Ind}_G^{P\#Q} \tau \to \operatorname{Ind}_L^P \eta$ *given by*

$$\varphi_\sigma(f)(q) = \widetilde{\sigma}_p^{-1}\big(f\big([p,q]\big)\big) \quad \textit{and} \quad \varphi_\tau(f)(p) = \widetilde{\tau}_q\big(f\big([p,q]\big)\big),$$

where $p \in P$ *and* $q \in Q$ *are such that* $\varphi(K \cdot p) = q \cdot G$. *These isomorphisms are equivariant and hence induce isomorphisms* $\Phi_\sigma := \varphi_\sigma \rtimes G$ *and* $\Phi_\tau := \varphi_\tau \rtimes K$ *of the crossed products.*

(2) *The maps defined by*

$$\psi_\sigma(f)(K \cdot p) := \widetilde{\sigma}_p^{-1}\big(f(p)\big) \quad \textit{and} \quad \psi_\tau(f)(q \cdot G) := \widetilde{\tau}_q\big(f(q)\big)$$

give isomorphisms $\psi_\sigma : \operatorname{Ind}_K^P \sigma \to C_0(K \backslash P, A)$ *and* $\psi_\tau : \operatorname{Ind}_G^Q \tau \to C_0(Q/G, A)$. *Furthermore, the composition*

$$\operatorname{Ind}_K^P \sigma \xrightarrow{\psi_\sigma} C_0(K \backslash P, A) \xrightarrow{\varphi_*^{-1}} C_0(Q/G, A) \xrightarrow{\psi_\tau^{-1}} \operatorname{Ind}_G^Q \tau, \qquad T : \operatorname{Ind}_K^P \sigma \to \operatorname{Ind}_G^Q \tau$$

is given by

$$T(f)(q) = \widetilde{\tau}_q^{-1}\widetilde{\sigma}_p^{-1}\big(f(p)\big), \tag{3.10}$$

where $p \in P$ *is such that* $\varphi(K \cdot p) = q \cdot G$. T *is equivariant, and hence* $\Phi := T \rtimes L$ *is an isomorphism of* $\operatorname{Ind}_K^P \sigma \rtimes_{\eta\otimes\mathrm{rt}} L$ *onto* $\operatorname{Ind}_G^Q \tau \rtimes_{\zeta\otimes\mathrm{lt}} L$.

Proof. (1) The first step is to verify that φ_σ is well defined. Let $f \in \operatorname{Ind}_K^{P\#Q} \sigma$. If $\varphi(K \cdot p) = q \cdot G$, then for any $k \in K$,

$$\widetilde{\sigma}_{k\cdot p}\big(f\big([k \cdot p, q]\big)\big) = (\sigma_k \widetilde{\sigma}_p)^{-1}\big(\sigma_k\big(f\big([p,q]\big)\big)\big) = \widetilde{\sigma}_p^{-1}\big(f\big([p,q]\big)\big).$$

It follows that $\varphi_\sigma(f)$ is a well-defined function on Q. On the other hand, if $\varphi(K \cdot p) = q \cdot G$, then, for all $t \in L$, $\varphi(K \cdot p \cdot t^{-1}) = t \cdot q \cdot G$ and

$$\begin{aligned}\varphi_\sigma(f)(t \cdot q) &= \widetilde{\sigma}_{p\cdot t^{-1}}^{-1}\big(f\big(\big[p \cdot t^{-1}, t \cdot q\big]\big)\big) = \widetilde{\sigma}_{p\cdot t^{-1}}^{-1}\big(f\big([p,q]\big)\big)\\ &= \zeta_t \widetilde{\sigma}_p^{-1}\big(f\big([p,q]\big)\big) = \zeta_t\big(\varphi_\sigma(f)(q)\big).\end{aligned}$$

Therefore, to see that $\varphi_\sigma(f)$ is in $\operatorname{Ind}_L^Q \zeta$, we only have to check that $\varphi_\sigma(f)$ is continuous and that $L \cdot q \mapsto \|f(q)\|$ vanishes at infinity.

To establish continuity, it suffices to show that, given any net $q_\alpha \to q$ we can find a subnet such that, after we pass to the subnet and relabel, we have $\varphi_\sigma(f)(q_\alpha) \to \varphi_\sigma(f)(q)$. Choose p_α such that $\varphi(K \cdot p_\alpha) = q_\alpha \cdot G$. Since φ is a homeomorphism, there is a p such that

$$K \cdot p_\alpha \to K \cdot p = \varphi^{-1}(q \cdot G).$$

Since the orbit map is open, we can pass to a subnet, relabel, and assume that there are $k_\alpha \in K$ such that

$$k_\alpha \cdot p_\alpha \to p.$$

Of course, $\varphi(Kk_\alpha \cdot p_\alpha) = q_\alpha \cdot G$, and

$$\varphi_\sigma(f)(q_\alpha) = \widetilde{\sigma}^{-1}_{k_\alpha \cdot p_\alpha}\bigl(f\bigl([k_\alpha \cdot p_\alpha, q_\alpha]\bigr)\bigr) \to \widetilde{\sigma}^{-1}_p\bigl(f\bigl([p,q]\bigr)\bigr) = \varphi_\sigma(f)(q)$$

because f and $\widetilde{\sigma}$ are continuous. Thus, $\varphi_\sigma(f)$ is continuous.

To see that $\varphi_\sigma(f)$ vanishes at infinity, it suffices to show that if $\{q_\alpha\}$ is a net in Q such that

$$\bigl\|\varphi_\sigma(f)(q_\alpha)\bigr\| \geqslant \epsilon > 0,$$

then $\{q_\alpha\}$ has a convergent subnet. Let p_α be such that $\varphi(K \cdot p_\alpha) = q_\alpha \cdot G$. Since $\widetilde{\sigma}^{-1}_{p_\alpha}$ is isometric, we must have

$$\bigl\|f\bigl([p_\alpha, q_\alpha]\bigr)\bigr\| \geqslant \epsilon \quad \text{for all } \alpha.$$

Then, since $K \cdot [p,q] \mapsto \|f([p,q])\|$ vanishes at infinity, we can pass to a subnet, relabel, and assume that there is a $[p,q] \in P \# Q$ such that

$$K \cdot [p_\alpha, q_\alpha] \to K \cdot [p,q].$$

Since orbit maps are open, we can pass to another subnet, relabel, and find $k_\alpha \in K$ such that

$$k_\alpha \cdot [p_\alpha, q_\alpha] = [k_\alpha \cdot p_\alpha, q_\alpha] \to [p,q].$$

Similarly, after passing to another subnet and relabeling, there are $t_\alpha \in L$ such that

$$\bigl(k_\alpha \cdot p_\alpha \cdot t_\alpha, t_\alpha^{-1} \cdot q_\alpha\bigr) \to (p,q).$$

In particular, $K \cdot q_\alpha \to K \cdot q$, and hence $\varphi_\sigma(f) \in \operatorname{Ind}_L^Q \zeta$. Since the operations are pointwise, φ_σ is a homomorphism of $\operatorname{Ind}_K^{P\#Q} \sigma$ into $\operatorname{Ind}_L^Q \zeta$; it is an isomorphism since similar considerations show that

$$\varphi_\sigma^{-1}(g)\bigl([p,q]\bigr) = \widetilde{\sigma}_p\bigl(g(q)\bigr)$$

is an inverse.

Furthermore, if $m \in G$, and if $\varphi(K \cdot p) = q \cdot G$, then $\varphi(K \cdot p) = (q \cdot m) \cdot G$, and, since τ and $\widetilde{\sigma}$ commute,

$$\begin{aligned}
\widetilde{\sigma}_p^{-1}\bigl(\tau_m \otimes \mathrm{rt}_m(f)\bigl([p,q]\bigr)\bigr) &= \widetilde{\sigma}_p^{-1}\bigl(\tau_m\bigl(f\bigl([p,qm]\bigr)\bigr)\bigr) \\
&= \tau_m\bigl(\widetilde{\sigma}_p^{-1}\bigl(f\bigl([p,qm]\bigr)\bigr)\bigr) \\
&= \tau_m\bigl(\varphi_\sigma(f)(q \cdot m)\bigr) \\
&= (\tau_m \otimes \mathrm{rt}_m)\varphi_\sigma(f)(q).
\end{aligned}$$

Thus φ_σ is equivariant, and therefore gives an isomorphism $\Phi_\sigma = \varphi_\sigma \rtimes G$.

The statements for φ_τ and Φ_τ are proved similarly.

(2) It is easy to check that ψ_σ and ψ_τ are well-defined homomorphisms which are isomorphisms by computing their inverses directly (for example, $\psi_\tau^{-1}(g)(q) = \widetilde{\tau}_q^{-1}(g(q \cdot G))$), and it is then straightforward to verify (3.10). Further, if $\varphi(K \cdot p) = q \cdot G$ and if $t \in L$, then on the one hand

$$T\big(\eta_t \otimes \mathrm{rt}_t(f)\big)(q) = \widetilde{\tau}_q^{-1}\widetilde{\sigma}_p^{-1}\big(\eta_t \otimes \mathrm{rt}_t(f)(p)\big) = \widetilde{\tau}_q^{-1}\widetilde{\sigma}_p^{-1}\big(\eta_t\big(f(p \cdot t)\big)\big). \tag{3.11}$$

On the other hand, we also have $\varphi(K \cdot p \cdot t) = t^{-1} \cdot q \cdot G$, and

$$\begin{aligned}\zeta_t \otimes \mathrm{lt}_t\big(T(f)\big)(q) &= \zeta_t\big(T(f)\big(t^{-1} \cdot q\big)\big)\\ &= \zeta_t\big(\widetilde{\tau}_{t^{-1}\cdot q}^{-1}\widetilde{\sigma}_{p\cdot t}^{-1}\big(f(p \cdot t)\big)\big)\end{aligned}$$

which, since $\widetilde{\tau}_{t^{-1}\cdot q} = \eta_t^{-1}\widetilde{\tau}_q$ and $\sigma_{p\cdot t} = \widetilde{\sigma}_p\zeta_t$, is

$$= \zeta_t\big(\widetilde{\tau}_q^{-1}\eta_t\zeta_t^{-1}\widetilde{\sigma}_p\big(f(p \cdot t)\big)\big),$$

and this coincides with (3.11) because ζ commutes with $\widetilde{\tau}$ and η, and η commutes with $\widetilde{\sigma}$. Thus, T is equivariant and the result follows. □

Proof of Theorem 3.1. Let Φ_σ, Φ_τ and Φ be as in Lemma 3.2. For fixed $x \in C_c(P, A)$, $y \in C_c(Q, A)$ and $(p, q) \in P \times Q$ set

$$f(p, q) := \int_L \widetilde{\tau}_{r^{-1}\cdot q}^{-1}\big(x(p \cdot r)\big)\widetilde{\sigma}_{p\cdot r}\big(y\big(r^{-1} \cdot q\big)\big)\,dr. \tag{3.12}$$

Straightforward computation using the left-invariance of Haar measure shows that $f(p, q)$ depends only on the class $[p, q]$ of $(p, q) \in P \# Q$. Since the actions of L on P and Q are free and proper, $f(p, q) < \infty$ and $[p, q] \mapsto f(p, q)$ is continuous with compact support. Thus we can define $\Omega : C_c(P, A) \odot C_c(Q, A) \to C_c(P \# Q, A)$ by letting

$$\Omega(x \otimes y)\big([p, q]\big) = f(p, q).$$

(That Ω is well defined on the balanced tensor product will follow from the same calculation that shows Ω is isometric for the right inner products; see below.)

To see that (3.9) commutes, we will show that the triple $(\Phi_\tau^{-1}, \Omega, \Phi_\sigma^{-1})$ extends to an imprimitivity bimodule isomorphism of $W(P) \otimes_\Phi W(Q)$ onto $W(P\#Q)$. In particular, we will show that $(\Phi_\tau^{-1}, \Omega, \Phi_\sigma^{-1})$ preserves the right inner products and both the left and right actions. Then the range of Ω will be a closed sub-bimodule of $W(P \# Q)$ on which the right inner product is full. It will then follow from the Rieffel correspondence (see, for example, [18, Proposition 3.24]) that Ω is surjective. This will imply that $(\Phi_\tau^{-1}, \Omega, \Phi_\sigma^{-1})$ must also preserve the left inner product and hence will be the desired isomorphism.

Let $x, w \in C_c(P, A) \subseteq W(P)$ and $y, z \in C_c(Q, A) \subseteq W(Q)$ and let $\langle\langle\cdot,\cdot\rangle\rangle_*$ be the right inner product on $W(P) \otimes_\Phi W(Q)$. We will show that

$$\langle\langle x \otimes y, w \otimes z\cdot\rangle\rangle_* = \Phi_\sigma\big(\big\langle \Omega(x \otimes y), \Omega(w \otimes z)\big\rangle_*\big).$$

The inner product $\langle\langle x \otimes y, w \otimes z\cdot\rangle\rangle_*$ takes values in $C_c(G, \mathrm{Ind}_K^Q \zeta) \subseteq \mathrm{Ind}_L^Q \zeta \rtimes G$ which we view as functions on $G \times Q$. Thus

$$\begin{aligned}
&\langle\langle x \otimes y, w \otimes z\rangle\rangle_*(m, q)\\
&\quad = \big\langle \Phi\big(\langle w, x\rangle_*\big) \cdot y, z\big\rangle_*(m, q)\\
&\quad = \Delta_G(m)^{1/2} \int_L \zeta_t\big(\big(\Phi\big(\langle w, x\rangle_*\big) \cdot y\big)\big(t^{-1} \cdot q\big)^* \tau_m\big(z\big(t^{-1} \cdot q \cdot m\big)\big)\big)\, dt\\
&\quad = \Delta_G(m)^{1/2} \int_L \zeta_t\bigg(\int_L \Phi\big(\langle w, x\rangle_*\big)\big(r, t^{-1} \cdot q\big)\zeta_r\big(y\big(r^{-1}t^{-1} \cdot q\big)\big)\Delta_L(r)^{1/2}\, dr\bigg)^*\\
&\qquad \zeta_t \tau_m\big(z\big(t^{-1} \cdot q \cdot m\big)\big)\, dt
\end{aligned}$$

which, if $\varphi(K \cdot p) = q \cdot G$, is

$$\begin{aligned}
&= \Delta_G(m)^{-1/2} \int_L \int_L \zeta_t\big(\widetilde{\tau}_{t^{-1}\cdot q}^{-1} \widetilde{\sigma}_{p\cdot t}^{-1}\big(\langle w, x\rangle_*(r, p \cdot t)\big)\zeta_r\big(y\big(r^{-1}t^{-1} \cdot q\big)\big)\big)^* \Delta_L(r)^{1/2}\, dr\\
&\quad \zeta_t \tau_m\big(z\big(t^{-1} \cdot q \cdot m\big)\big)\, dt\\
&= \Delta_G(m)^{-1/2} \int_L \int_L \zeta_{tr}\big(y\big(r^{-1}t^{-1} \cdot q\big)\big)^*\\
&\quad \zeta_t \widetilde{\tau}_{t^{-1}\cdot q}^{-1} \widetilde{\sigma}_{p\cdot t}^{-1}\bigg(\int_K \sigma_s\big(w\big(s^{-1} \cdot p \cdot t\big)^* \eta_r\big(x\big(s^{-1} \cdot p \cdot tr\big)\big)\big)\, ds\bigg)^* \zeta_t \tau_m\big(z\big(t^{-1} \cdot q \cdot m\big)\big)\, dr\, dt
\end{aligned}$$

which, since $\widetilde{\tau}_{t^{-1}\cdot q} = \eta_t^{-1}\widetilde{\tau}_q$ and $\widetilde{\sigma}_{p\cdot t} = \widetilde{\sigma}_p \zeta_t$, and since ζ commutes with both $\widetilde{\tau}$ and η, and η commutes with $\widetilde{\sigma}$ (see (3.3)–(3.5)), is

$$\begin{aligned}
&= \Delta_G(m)^{-1/2} \int_L \int_L \int_K \zeta_{tr}\big(y\big(r^{-1}t^{-1} \cdot q\big)^*\big)\\
&\quad \widetilde{\tau}_q^{-1} \widetilde{\sigma}_p^{-1}\big(\sigma_s \eta_{tr}\big(x\big(s^{-1} \cdot p \cdot tr\big)^*\big)\eta_t \sigma_s\big(w\big(s^{-1} \cdot p \cdot t\big)\big)\big)\zeta_t \tau_m\big(z\big(t^{-1} \cdot q \cdot m\big)\big)\, ds\, dr\, dt
\end{aligned}$$

which, replacing r by $t^{-1}r$ and using (3.3)–(3.5) again, is

$$= \Delta_G(m)^{-1/2} \int_K \bigg(\int_L \zeta_r\big(y\big(r^{-1} \cdot q\big)^*\big)\widetilde{\tau}_{r^{-1}\cdot q}^{-1} \widetilde{\sigma}_{s^{-1}\cdot p}^{-1}\big(x\big(s^{-1} \cdot p \cdot r\big)^*\big)\bigg)$$

$$\left(\int_L \widetilde{\tau}^{-1}_{t^{-1}\cdot q}\widetilde{\sigma}^{-1}_{s^{-1}\cdot p}\big(w\big(s^{-1}\cdot p\cdot t\big)\big)\zeta_t\tau_m\big(z\big(t^{-1}\cdot q\cdot m\big)\big)\,dt\right)ds$$

which, using (3.3), is

$$= \Delta_G(m)^{-1/2}\int_K \widetilde{\sigma}^{-1}_{s^{-1}\cdot p}\left(\left(\left(\int_L \widetilde{\tau}^{-1}_{r^{-1}\cdot q}\big(x\big(s^{-1}\cdot p\cdot r\big)\big)\widetilde{\sigma}_{s^{-1}\cdot p\cdot r}\big(y\big(r^{-1}\cdot q\big)\big)\,dr\right)^*\right.\right.$$
$$\left.\left.\left(\int_L \widetilde{\tau}^{-1}_{t^{-1}\cdot q}\big(w\big(s^{-1}\cdot p\cdot t\big)\big)\widetilde{\sigma}_{s^{-1}\cdot p\cdot t}\tau_m\big(z\big(t^{-1}\cdot q\cdot m\big)\big)\,dt\right)\right)\right)ds$$

which, using (3.5) and the definition of Ω, is

$$= \Delta_G(m)^{-1/2}\int_K \widetilde{\sigma}^{-1}_{s^{-1}\cdot p}\left(\Omega(x\otimes y)\big(\big[s^{-1}\cdot p, q\big]\big)^*\right.$$
$$\left.\tau_m\left(\int_L \widetilde{\tau}^{-1}_{t^{-1}\cdot q\cdot m}\big(w\big(s^{-1}\cdot p\cdot t\big)\big)\widetilde{\sigma}_{s^{-1}\cdot p\cdot t}\big(z\big(t^{-1}\cdot q\cdot m\big)\big)\,dt\right)\right)ds$$
$$= \Delta_G(m)^{-1/2}\widetilde{\sigma}_p^{-1}\left(\int_K \sigma_s\big(\Omega(x\otimes y)\big(\big[s^{-1}\cdot p, q\big]\big)^*\tau_m\big(\Omega(w\otimes z)\big(\big[s^{-1}\cdot p, q\cdot m\big]\big)\big)\,ds\big)\right)$$
$$= \widetilde{\sigma}_p^{-1}\big(\big\langle\Omega(x\otimes y), \Omega(w\otimes z)\big\rangle_*\big(m,[p,q]\big)\big)$$
$$= \Phi_\sigma\big(\big\langle\Omega(x\otimes y), \Omega(w\otimes z)\big\rangle_*\big)(m,q).$$

Thus $(\Phi_\tau^{-1}, \Omega, \Phi_\sigma^{-1})$ intertwines the right inner products.

If $b\in C_c(G, \operatorname{Ind}_K^{P\#Q}\sigma)\subseteq \operatorname{Ind}_K^{P\#Q}\sigma\rtimes_{\tau\otimes\mathrm{rt}} G$ is viewed as a function on $G\times(P\# Q)$, then

$$\Omega(x\otimes y)\cdot b\big([p,q]\big)$$
$$= \int_G \tau_m\big(\Omega(x\otimes y)\big([p,q\cdot m]\big)\big)b\big(m^{-1},[p,q\cdot m]\big)\Delta_G(m)^{-1/2}\,dm$$
$$= \int_G \tau_m\left(\int_L \widetilde{\tau}^{-1}_{r^{-1}\cdot q\cdot m}\big(x(p\cdot r)\big)\widetilde{\sigma}_{p\cdot r}\big(y\big(r^{-1}\cdot q\cdot m\big)\big)\,dr\right)b\big(m^{-1},[p,q\cdot m]\big)\Delta_G(m)^{-1/2}\,dm$$
$$= \int_G\int_L \widetilde{\tau}^{-1}_{r^{-1}\cdot q}\big(x(p\cdot r)\big)\tau_m\widetilde{\sigma}_{p\cdot r}\big(y\big(r^{-1}\cdot q\cdot m\big)\big)b\big(m^{-1},[p,q\cdot m]\big)\,dr\,\Delta_G(m)^{-1/2}\,dm.$$

On the other hand,

$$\Omega\big((x\otimes y)\cdot\Phi_\sigma(b)\big)\big([p,q]\big)$$
$$= \Omega\big(x\otimes\big(y\cdot\Phi_\sigma(b)\big)\big)\big([p,q]\big)$$

$$= \int_L \widetilde{\tau}^{-1}_{r^{-1}\cdot q}\big(x(p\cdot r)\big)\widetilde{\sigma}_{p\cdot r}\big(y\cdot \Phi_\sigma(b)\big(r^{-1}\cdot q\big)\big)\,dr$$

$$= \int_L \widetilde{\tau}^{-1}_{r^{-1}\cdot q}\big(x(p\cdot r)\big)\widetilde{\sigma}_{p\cdot r}\bigg(\int_G \tau_m\big(y\big(r^{-1}\cdot q\cdot m\big)\Phi_\sigma(b)\big(m^{-1}, r^{-1}\cdot q\cdot m\big)\big)\Delta_G(m)^{-1/2}\,dm\bigg)\,dr$$

$$= \int_L\int_G \widetilde{\tau}^{-1}_{r^{-1}\cdot q}\big(x(p\cdot r)\big)\widetilde{\sigma}_{p\cdot r}\tau_m\big(y\big(r^{-1}\cdot q\cdot m\big)\big)b\big(m^{-1}, [p, q\cdot m]\big)\Delta_G(m)^{-1/2}\,dm\,dr$$

where we have used that $\varphi(K\cdot p\cdot r) = r^{-1}\cdot q\cdot G$ implies

$$\Phi_\sigma(b)\big(m^{-1}, r^{-1}\cdot q\cdot m\big) = \widetilde{\sigma}^{-1}_{p\cdot r}\big(b\big(m^{-1}, \big[p\cdot r, r^{-1}\cdot q\cdot m\big]\big)\big) = \widetilde{\sigma}^{-1}_{p\cdot r}\big(b\big(m^{-1}, [p, q\cdot m]\big)\big).$$

Since τ and $\widetilde{\sigma}$ commute, an application of Fubini's theorem gives $\Omega(x\otimes y)\cdot b = \Omega((x\otimes y)\cdot \Phi_\sigma(b))$.

For the left action, let $c \in C_c(K, \operatorname{Ind}_G^{P\#Q}\tau) \subseteq \operatorname{Ind}_G^{P\#Q}\alpha \rtimes_{\sigma\otimes \mathrm{lt}} K$. Then, viewing c as a function on $K\times(P\,\#\,Q)$, we have

$$\begin{aligned}
&c\cdot \Omega(x\otimes y)\big([p,q]\big)\\
&\quad= \int_K c\big(t,[p,q]\big)\sigma_t\big(\Omega(x\otimes y)\big(\big[t^{-1}\cdot p, q\big]\big)\big)\Delta_K(t)^{1/2}\,dt\\
&\quad= \int_K c\big(t,[p,q]\big)\sigma_t\bigg(\int_L \widetilde{\tau}^{-1}_{r^{-1}\cdot q}\big(x\big(t^{-1}\cdot p\cdot r\big)\big)\widetilde{\sigma}_{t^{-1}\cdot p\cdot r}\big(y\big(r^{-1}\cdot q\big)\big)\,dr\bigg)\Delta_K(t)^{1/2}\,dt\\
&\quad= \int_K\int_L c\big(t,[p,q]\big)\sigma_t\widetilde{\tau}^{-1}_{r^{-1}\cdot q}\big(x\big(t^{-1}\cdot p\cdot r\big)\big)\widetilde{\sigma}_{p\cdot r}\big(y\big(r^{-1}\cdot q\big)\big)\Delta_K(t)^{1/2}\,dr\,dt. \qquad (3.13)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\Omega\big(\Phi_\tau(c)\cdot(x\otimes y)\big)\big([p,q]\big)\\
&\quad= \Omega\big(\Phi_\tau(c)\cdot x\otimes y\big)\big([p,q]\big)\\
&\quad= \int_L \widetilde{\tau}^{-1}_{r^{-1}\cdot q}\big(\Phi_\tau(c)\cdot x(p\cdot r)\big)\widetilde{\sigma}_{p\cdot r}\big(y\big(r^{-1}\cdot q\big)\big)\,dr\\
&\quad= \int_L \widetilde{\tau}^{-1}_{r^{-1}\cdot q}\bigg(\int_K \Phi_\tau(c)(t, p\cdot r)\sigma_t\big(x\big(t^{-1}\cdot p\cdot r\big)\big)\Delta_K(t)^{1/2}\,dt\bigg)\widetilde{\sigma}_{p\cdot r}\big(y\big(r^{-1}\cdot q\big)\big)\,dr\\
&\quad= \int_L \widetilde{\tau}^{-1}_{r^{-1}\cdot q}\bigg(\int_K \widetilde{\tau}_{r^{-1}\cdot q}\big(c\big(t,[p,q]\big)\big)\sigma_t\big(x\big(t^{-1}\cdot p\cdot r\big)\big)\Delta_K(t)^{1/2}\,dt\bigg)\widetilde{\sigma}_{p\cdot r}\big(y\big(r^{-1}\cdot q\big)\big)\,dr
\end{aligned}$$

$$= \int\limits_L \int\limits_G c\big(t, [p,q]\big)\widetilde{\tau}^{-1}_{r^{-1}\cdot q}\sigma_t\big(x\big(t^{-1}\cdot p\cdot r\big)\big)\widetilde{\sigma}_{p\cdot r}\big(y\big(r^{-1}\cdot q\big)\big)\Delta_K(t)^{1/2}\,dt\,dr,$$

which coincides with (3.13). This completes the proof. □

4. Imprimitivity bimodule isomorphisms

In this section, we show that for a dual coaction $\hat{\alpha}$, the Mansfield bimodule $Y^{G/N}_{G/H}(\hat{\alpha}|)$ appearing in Theorem 1.1 can be replaced by a symmetric imprimitivity bimodule. More precisely, we show how this result (Theorem 4.1) follows from a certain bimodule factorization result (Theorem 4.2). Preparation for the proof of Theorem 4.2 takes up the rest of this section; the proof itself occupies Section 5.

Theorem 4.1. *Suppose α is a continuous action of a locally compact group G by automorphisms of a C^*-algebra A, and suppose N and H are closed normal subgroups of G with $N \subseteq H$. Let ϵ be the maximal coaction $\hat{\alpha}|_{G/N}$ of G/N on $A \rtimes_\alpha G$. Then the diagram*

$$\begin{CD}
A \rtimes_\alpha G \rtimes_\epsilon (G/N) \rtimes_{\hat{\epsilon}|} (H/N) @>{Y^{G/N}_{G/H}(\epsilon)}>> A \rtimes_\alpha G \rtimes_{\epsilon|} (G/H) \\
@V{\cong}VV @VV{\cong}V \\
(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes \mathrm{lt}} G \rtimes_{\beta|} (H/N) @>{Z^{G/N}_{G/H}(\alpha)}>> (A \otimes C_0(G/H)) \rtimes_{\alpha\otimes \mathrm{lt}} G
\end{CD} \tag{4.1}$$

of imprimitivity bimodules commutes, where the vertical arrows are the canonical isomorphisms.

Here $Z^{G/N}_{G/H}(\alpha)$ is the symmetric-imprimitivity bimodule constructed in [11, Proposition 3.3]; we will review its construction in Section 4.2. The action

$$\beta : G/N \to \mathrm{Aut}\big(\big(A \otimes C_0(G/N)\big) \rtimes_{\alpha\otimes\mathrm{lt}} G\big) \tag{4.2}$$

is induced by the action $\mathrm{id} \otimes \mathrm{rt}$ of G/N on $A \otimes C_0(G/N)$, which commutes with the action $\alpha \otimes \mathrm{lt}$ of G. It corresponds to the dual of the coaction $\epsilon = \hat{\alpha}|_{G/N}$ under the canonical isomorphism of $A \rtimes_\alpha G \rtimes_{\hat{\alpha}|} (G/N)$ with $(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G$.

Dual coactions and their restrictions are maximal by [5, Proposition 3.4] and [13, Corollary 7.2], so ϵ is a maximal coaction of G/N on $A \rtimes_\alpha G$. Thus, the Katayama bimodule $K(\epsilon)$ (see the discussion following (2.1)) is an $(A \rtimes_\alpha G) \rtimes_\epsilon G/N \rtimes_{\hat{\epsilon}} G/N$–$A \rtimes_\alpha G$ imprimitivity bimodule. By [13, Proposition 4.2], $K(\epsilon)$ comes equipped with a $\hat{\hat{\epsilon}}$–ϵ compatible coaction ϵ_K, and we can further restrict these coactions to G/H and take crossed products (see, for example, [6, §3.1.2]).

The following factorization theorem for $Z^{G/N}_{G/H}(\alpha)$ generalises [13, Proposition 6.3], which is the one-subgroup version.

Theorem 4.2. *Under the hypotheses of Theorem* 4.1*, the diagram*

$$\begin{array}{ccc}
(A\otimes C_0(G/N))\rtimes_{\alpha\otimes\mathrm{lt}} G\rtimes_{\beta|}(H/N) & \xrightarrow{Z^{G/N}_{G/H}(\alpha)} & (A\otimes C_0(G/H))\rtimes_{\alpha\otimes\mathrm{lt}} G \\
\cong\Big\uparrow & & \cong\Big\uparrow \\
A\rtimes_\alpha G\rtimes_\epsilon (G/N)\rtimes_{\hat\epsilon|}(H/N) & & A\rtimes_\alpha G\rtimes_{\epsilon|}(G/H) \\
X^{G/N}_{H/N}(\hat\epsilon)\Big\uparrow & & \Big\uparrow K(\epsilon)\rtimes(G/H) \\
((A\rtimes_\alpha G\rtimes_\epsilon (G/N))\otimes C_0(G/H))\underset{\hat\epsilon\otimes\mathrm{lt}}{\rtimes}(G/N) & \overset{\cong}{\longleftarrow} & A\rtimes_\alpha G\rtimes_\epsilon (G/N)\rtimes_{\hat\epsilon}(G/N)\rtimes_{\hat{\hat\epsilon}|}(G/H)
\end{array} \tag{4.3}$$

of imprimitivity bimodules commutes, where all the isomorphisms are canonical.

We now show that Theorem 4.1 follows from Theorem 4.2. See Section 5 for the proof of Theorem 4.2.

Proof of Theorem 4.1. Insert the arrow

$$A\rtimes_\alpha G\rtimes_\epsilon (G/N)\rtimes_{\hat\epsilon|}(H/N)\xrightarrow{Y^{G/N}_{G/H}(\epsilon)} A\rtimes_\alpha G\rtimes_{\epsilon|}(G/H)$$

into the middle of commutative diagram (4.3) to create an upper square and a lower square. Applying Corollary 6.4 of [13] to the maximal coaction $\epsilon=\hat\alpha|$ of G/N on $A\rtimes_\alpha G$ shows that the lower square commutes; since all arrows are invertible, it follows that the upper square—which is precisely (4.1)—commutes as well. □

In the remainder of this section, we prepare for the proof of Theorem 4.2 by identifying the three bimodules in (4.3) with symmetric-imprimitivity bimodules. We retain the notation and hypotheses used thus far in this section (but we will carefully note situations where in fact H need not be normal in G).

4.1. *Realising $X^{G/N}_{H/N}(\hat\epsilon)$ as a symmetric-imprimitivity bimodule*

It is well known how to use the symmetric imprimitivity theorem to derive Green's imprimitivity theorem (Proposition 3 in [7]). It turns out that, for the action β of G/N on $(A\otimes C_0(G/N))\rtimes_{\alpha\otimes\mathrm{lt}} G$ as at (4.2) and the subgroup $H/N\subseteq G/N$, the symmetric imprimitivity theorem can produce the Green bimodule from a somewhat different set-up. In this subsection, H need not be normal in G.

First note that the identity map on $C_c(H/N\times G\times G/N, A)$ extends to an isomorphism

$$i:\big(\big(A\otimes C_0(G/N)\big)\rtimes_{\alpha\otimes\mathrm{lt}} G\big)\rtimes_{\beta|}(H/N)\to\big(A\otimes C_0(G/N)\big)\rtimes_\gamma (H/N\times G),$$

where $\gamma=(\mathrm{id}\times\alpha)\otimes(\mathrm{rt}\times\mathrm{lt})=(\mathrm{id}\otimes\mathrm{rt})\times(\alpha\otimes\mathrm{lt})$. The map

$$\iota: C_c(G/N\times G/H\times G\times G/N, A)\to C_c(G/N\times G\times G/N\times G/H, A)$$

defined by $\iota(g)(tN, s, rN, uH) = g(tN, uH, s, rN)$ extends to an isomorphism

$$\iota : \big(\big(A \otimes C_0(G/N) \rtimes_{\alpha\otimes\mathrm{lt}} G\big) \otimes C_0(G/H)\big) \rtimes_{\beta\otimes\mathrm{lt}} (G/N)$$
$$\to \big(A \otimes C_0(G/N) \otimes C_0(G/H)\big) \rtimes_{\upsilon} (G/N \times G),$$

where υ is the action $(\mathrm{id} \times \alpha) \otimes (\mathrm{rt} \times \mathrm{lt}) \otimes (\mathrm{lt} \times \mathrm{id}) = (\mathrm{id} \otimes \mathrm{rt} \otimes \mathrm{lt}) \times (\alpha \otimes \mathrm{lt} \otimes \mathrm{id})$.

Now consider the symmetric imprimitivity data $({}_K P_L, A, \sigma, \eta)$ defined as follows:

$$\begin{array}{ccc} & P = G/N \times G \times G/N & \\ {\scriptstyle (tN,s)\cdot(rN,u,vN)=(trN,su,tvN)} \nearrow & & \nwarrow {\scriptstyle (rN,u,vN)\cdot(hN,y)=(ryN,uy,vhN)} \\ K = G/N \times G & & L = H/N \times G \\ {\scriptstyle \sigma=\mathrm{id}\times\alpha} \searrow & & \swarrow {\scriptstyle \eta=\mathrm{id}} \\ & A & \end{array} \tag{4.4}$$

The symmetric imprimitivity theorem gives an $(\mathrm{Ind}_L^P \eta \rtimes_{\sigma\otimes\mathrm{lt}} K)$–$(\mathrm{Ind}_K^P \sigma \rtimes_{\eta\otimes\mathrm{rt}} L)$ imprimitivity bimodule $W(P) = W({}_K P_L, A, \sigma, \eta)$.

Proposition 4.3. *Suppose $\alpha : G \to \mathrm{Aut}\, A$ is a continuous action of a locally compact group G by automorphisms of a C^*-algebra A, and suppose N and H are closed subgroups of G with N normal in G and $N \subseteq H$. Let $X_{H/N}^{G/N}(\beta)$ be the Green bimodule associated to the action β of G/N on $(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G$ described at* (4.2)*, and let $W(P)$ be the imprimitivity bimodule associated to the symmetric imprimitivity data $({}_K P_L, A, \sigma, \eta)$ described at* (4.4)*. Then there are* (*noncanonical*) *equivariant isomorphisms $\Gamma : A \otimes C_0(G/N) \otimes C_0(G/H) \to \mathrm{Ind}_L^P \eta$ and $\Upsilon : A \otimes C_0(G/N) \to \mathrm{Ind}_K^P \sigma$ such that the diagram*

$$\begin{array}{ccc} (((A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G) \otimes C_0(G/H)) \rtimes_{\beta\otimes\mathrm{lt}} (G/N) & \xrightarrow{X_{H/N}^{G/N}(\beta)} & ((A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G) \rtimes_{\beta|} (H/N) \\ {\scriptstyle (\Gamma\rtimes K)\circ\iota} \downarrow \cong & & \cong \downarrow {\scriptstyle (\Upsilon\rtimes L)\circ i} \\ \mathrm{Ind}_L^P \eta \rtimes_{\sigma\otimes\mathrm{lt}} K & \xrightarrow{W(P)} & \mathrm{Ind}_K^P \sigma \rtimes_{\eta\otimes\mathrm{rt}} L \end{array} \tag{4.5}$$

of imprimitivity bimodules commutes.

Proof. For $f \in A \otimes C_0(G/N) \cong C_0(G/N, A)$ and $(rN, u, vN) \in G/N \times G \times G/N$ define

$$\Upsilon(f)(rN, u, vN) = \alpha_u\big(f\big(r^{-1}vN\big)\big). \tag{4.6}$$

Then, for $(tN, s) \in K = G/N \times G$,

$$\begin{aligned} \Upsilon(f)\big((tN, s)\cdot(rN, u, vN)\big) &= \Upsilon(f)(trN, su, tvN) = \alpha_{su}\big(f\big(r^{-1}vN\big)\big) \\ &= \alpha_s\big(\Upsilon(f)(rN, u, vN)\big) = \sigma_{(tN,s)}\big(\Upsilon(f)(rN, u, vN)\big), \end{aligned}$$

so Υ maps $A \otimes C_0(G/N)$ into $\operatorname{Ind}_K^P \sigma$. It is straightforward to check that Υ is invertible, with inverse given by $\Upsilon^{-1}(g)(tN) = g(N, e, tN)$. For $(hN, y) \in L = N/H \times G$,

$$\begin{aligned}\Upsilon\big(\gamma_{(hN,y)}(f)\big)(rN, u, vN) &= \alpha_u\big(\gamma_{(hN,y)}(f)\big(r^{-1}vN\big)\big)\\ &= \alpha_{uy}\big(f\big(y^{-1}r^{-1}vhN\big)\big)\\ &= \Upsilon(f)(ryN, uy, vhN)\\ &= \Upsilon(f)\big((rN, u, vN)\cdot(hN, y)\big)\\ &= (\eta \otimes \mathrm{rt})_{(hN,y)}\big(\Upsilon(f)\big)(rN, u, vN),\end{aligned}$$

so Υ is a γ–$(\eta \otimes \mathrm{rt})$ equivariant isomorphism and induces an isomorphism

$$\Upsilon \rtimes L : \big(A \otimes C_0(G/N)\big) \rtimes_\gamma L \to \operatorname{Ind}_K^P \sigma \rtimes_{\eta\otimes\mathrm{rt}} L$$

of the crossed products.

Similarly, the map $\Gamma : A \otimes C_0(G/N) \otimes C_0(G/H) \to \operatorname{Ind}_L^P \eta$ defined by

$$\Gamma(f)(rN, u, vN) = f\big(ur^{-1}N, vH\big) \tag{4.7}$$

is an ϵ–$(\sigma \otimes \mathrm{lt})$ equivariant isomorphism with inverse given by $\Gamma^{-1}(g)(vN, rH) = g(v^{-1}N, e, rN)$. So Γ induces an isomorphism

$$\Gamma \rtimes K : \big(A \otimes C_0(G/N) \otimes C_0(G/H)\big) \rtimes_\epsilon K \to \operatorname{Ind}_L^P \eta \rtimes_{\sigma\otimes\mathrm{lt}} K.$$

Let $\Psi : C_c(G/N \times G \times G/N, A) \to C_c(G/N \times G \times G/N, A)$ be the map

$$\Psi(rN, u, vN) = f\big(vN, u, ur^{-1}N\big)\Delta_G(u)^{1/2}.$$

We will show that the triple $((\Gamma \rtimes K)\circ\iota, \Psi, (\Upsilon \rtimes L)\circ i)$ extends to an imprimitivity bimodule isomorphism of $X_{H/N}^{G/N}(\beta)$ onto $W(P)$. We may view both $X_{H/N}^{G/N}(\beta)$ and $W(P)$ as completions of $C_c(G/N \times G \times G/N, A)$, so Ψ clearly has dense range. It therefore suffices to show, for $x, y \in C_c(G/N \times G \times G/N, A) \subseteq X_{H/N}^{G/N}(\beta)$ and $f \in C_c(H/N \times G \times G/N, A)$, that

(1) $\Psi(x \cdot f) = \Psi(x)\cdot((\Upsilon \rtimes L)\circ i(f))$ and
(2) $(\Gamma \rtimes K)\circ\iota({}_*\langle x, y\rangle) = {}_*\langle\Psi(x), \Psi(y)\rangle$.

(For then (2) implies

$$\big\|\Psi(g\cdot x) - \big((\Gamma \rtimes K)\circ\iota(g)\big)\cdot\Psi(x)\big\|^2 = 0$$

for $g \in C_c(G/N \times G/H \times G \times G/N, A)$, and this together with (1), (2), and denseness gives the other inner product condition.)

So let x, y and f be as above. Using the formula for the right action in Green's bimodule from [6, Eq. B.5] we have:

$$\begin{aligned}
&\Psi(x\cdot f)(rN,u,vN)\\
&\quad=\Delta_G(u)^{1/2}(x\cdot f)\big(vN,u,ur^{-1}N\big)\\
&\quad=\Delta_G(u)^{1/2}\int\limits_{H/N} x(vhN,\cdot,\cdot)\beta_{vhN}\big(f\big(h^{-1}N,\cdot,\cdot\big)\big)\Delta_{H/N}(hN)^{-1/2}\,d(hN)\,\big(u,ur^{-1}N\big)\\
&\quad=\Delta_G(u)^{1/2}\int\limits_{H/N}\int\limits_G x(vhN,s,\cdot)(\alpha\otimes \mathrm{lt})_s\big(\beta_{vhN}\big(f\big(h^{-1}N,s^{-1}u,\cdot\big)\big)\big)\\
&\qquad\Delta_{H/N}(hN)^{-1/2}\,d(hN)\,ds\,\big(ur^{-1}N\big)\\
&\quad=\Delta_G(u)^{1/2}\int\limits_{H/N}\int\limits_G x\big(vhN,s,ur^{-1}N\big)\alpha_s\big(f\big(h^{-1}N,s^{-1}u,s^{-1}ur^{-1}vhN\big)\big)\\
&\qquad\Delta_{H/N}(hN)^{-1/2}\,d(hN)\,ds.
\end{aligned}$$

Using the formula for the right action on $W(P)$ from [6, Eq. B.2] we have

$$\begin{aligned}
&\big(\Psi(x)\cdot\big((\Upsilon\rtimes L)\circ i(f)\big)\big)(rN,u,vN)\\
&\quad=\int\limits_{H/N\times G}\eta_{(hN,t)}\big(\Psi(x)\big((rN,u,vN)\cdot(hN,t)\big)\\
&\qquad\Upsilon\rtimes L(f)\big((hN,t)^{-1},(rN,u,vN)\cdot(hN,t)\big)\big)\Delta_{H/N\times G}\big((hN,t)\big)^{-1/2}\,d(hN,t)\\
&\quad=\int\limits_{H/N}\int\limits_G \Psi(x)(rtN,ut,vhN)\Upsilon\big(f\big(h^{-1}N,t^{-1}\big)\big)(rtN,utN,vhN)\\
&\qquad\Delta_{H/N}(hN)\Delta_G(t)^{-1/2}\,d(hN)\,dt\\
&\quad=\Delta_G(u)^{1/2}\int\limits_{H/N}\int\limits_G x\big(vhN,ut,ur^{-1}N\big)\alpha_{ut}\big(f\big(h^{-1}N,t^{-1},t^{-1}r^{-1}vhN\big)\big)\\
&\qquad\Delta_{H/N}(hN)^{-1/2}\,d(hN)\,dt,
\end{aligned}$$

which equals $\Psi(x\cdot f)(rN,u,vN)$ by the change of variable $s=ut$. Also,

$$\begin{aligned}
&(\Gamma\rtimes K)\circ\iota\big({}_*\langle x,y\rangle\big)\big((tN,s),(rN,u,vN)\big)\\
&\quad=\Gamma\big(\iota\big({}_*\langle x,y\rangle\big)(tN,s,\cdot,\cdot)\big)(rN,u,vN)\\
&\quad={}_*\langle x,y\rangle\big(tN,vH,s,ur^{-1}N\big)\\
&\quad=\Delta_{G/N}(tN)^{-1/2}\int\limits_{H/N} x(vhN,\cdot,\cdot)\beta_{tN}\big(y\big(t^{-1}vhN,\cdot,\cdot\big)^*\big)\,d(hN)\big(s,ur^{-1}N\big)
\end{aligned}$$

(note that the product $x(vhN,\cdot,\cdot)\beta_{tN}(y(t^{-1}vhN,\cdot,\cdot)^*)$ is convolution in $(A\otimes C_0(G/N))\rtimes_{\alpha\otimes\mathrm{lt}} G$)

$$= \Delta_{G/N}(tN)^{-1/2}\int_{H/N}\int_G x(vhN,w,\cdot)(\alpha\otimes\mathrm{lt})_w\big(\beta_{tN}\big(y\big(t^{-1}vhN,\cdot,\cdot\big)^*\big)\big)\big(w^{-1}s\big)\,dw\,d(hN)\,\big(ur^{-1}N\big)$$

$$= \Delta_{G/N}(tN)^{-1/2}\int_{H/N}\int_G x\big(vhN,w,ur^{-1}N\big)(\alpha\otimes\mathrm{lt})_w\big(y\big(t^{-1}vhN,\cdot,\cdot\big)^*\big)\big(w^{-1}s,ur^{-1}tN\big)\,dw\,d(hN)$$

(note that the involution $y(t^{-1}vhN,\cdot,\cdot)^*$ is in $(A\otimes C_0(G/N))\rtimes_{\alpha\otimes\mathrm{lt}} G$)

$$= \Delta_{G/N}(tN)^{-1/2}\int_{H/N}\int_G x(vhN,w,\cdot)(\alpha\otimes\mathrm{lt})_{ww^{-1}s}\big(y\big(t^{-1}vhN,s^{-1}w,\cdot\big)^*\big)\big(ur^{-1}tN\big)\,\Delta_G\big(s^{-1}w\big)\,dw\,d(hN)$$

$$= \Delta_{G/N}(tN)^{-1/2}\int_{H/N}\int_G x\big(vhN,w,ur^{-1}N\big)\alpha_s\big(y\big(t^{-1}vhN,s^{-1}w,s^{-1}ur^{-1}tN\big)^*\big)\,\Delta_G\big(s^{-1}w\big)\,dw\,d(hN)$$

$$= \Delta_{G/N}(tN)^{-1/2}\int_{H/N}\int_G x\big(vhN,uw,ur^{-1}N\big)\alpha_s\big(y\big(t^{-1}vh,s^{-1}uw,s^{-1}ur^{-1}tN\big)^*\big)\,\Delta_G\big(s^{-1}uw\big)\,d(hN)\,dw$$

$$= \int_{H/N}\int_G \Psi(x)(rwN,uw,vhN)\alpha_s\big(\Psi(y)\big(t^{-1}rwN,s^{-1}uw,t^{-1}vhN\big)^*\big)\,\Delta_{G/N}(tN)^{-1/2}\Delta_G(s)^{-1/2}\,d(hN)\,dw$$

$$= \int_{H/N\times G} \eta_{(hN,w)}\big(\Psi(x)\big((rN,u,vN)\cdot(hN,w)\big)\,\sigma_{(tN,s)}\big(\Psi(y)\big((tN,s)^{-1}(rN,u,vN)\cdot(hN,w)\big)^*\big)\big)\,\Delta_{G/N\times G}\big((tN,s)\big)^{-1/2}\,d(hN)\,dw$$

$$= {}_*\big\langle\Psi(x),\Psi(y)\big\rangle\big((tN,s),(rN,u,vN)\big). \qquad\square$$

4.2. The definition of $Z_{G/H}^{G/N}(\alpha)$

In [11], $Z_{G/H}^{G/N}(\alpha)$ is defined using symmetric imprimitivity data $({}_L Q_G, A, \zeta, \tau)$ as follows:

$$\begin{array}{ccccc} & & Q = G/N \times G & & \\ & \nearrow^{(hN,x)\cdot(wN,z)=(hwN,xz)} & & \nwarrow^{(wN,z)\cdot y=(wyN,zy)} & \\ L = H/N \times G & & & & G \\ & \searrow_{\zeta=\mathrm{id}\times\alpha} & & \swarrow_{\tau=\mathrm{id}} & \\ & & A & & \end{array} \tag{4.8}$$

(Here again, H need not be normal in G.) More precisely, the symmetric imprimitivity theorem gives an $(\mathrm{Ind}_G^Q \tau \rtimes_{\zeta\otimes\mathrm{lt}} L)$–$(\mathrm{Ind}_L^Q \zeta \rtimes_{\tau\otimes\mathrm{rt}} G)$ imprimitivity bimodule $W(Q) = W({}_L Q_G, A, \zeta, \tau)$. The map $\Omega : A \otimes C_0(G/H) \to \mathrm{Ind}_L^Q \zeta$ defined by

$$\Omega(f)(wN, z) = \alpha_z\big(f\big(w^{-1}H\big)\big) \tag{4.9}$$

is an $(\alpha\otimes\mathrm{lt})$–$(\tau\otimes\mathrm{rt})$ equivariant isomorphism with inverse given by $\Omega^{-1}(g)(tH) = g(t^{-1}N, e)$ and hence induces an isomorphism $\Omega \rtimes G$ of $(A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\mathrm{lt}} G$ onto $\mathrm{Ind}_L^Q \zeta \rtimes_{\tau\otimes\mathrm{rt}} G$. The map $\Theta : A \otimes C_0(G/N) \to \mathrm{Ind}_G^Q \tau$ defined by

$$\Theta(f)(wN, z) = f\big(zw^{-1}N\big) \quad \big((wN, z) \in Q = G/N \times G\big) \tag{4.10}$$

is a γ–$(\zeta \otimes \mathrm{lt})$ equivariant isomorphism with inverse given by $\Theta^{-1}(g)(rN) = g(r^{-1}N, e) = g(N, r)$. So Θ induces an isomorphism $\Theta \rtimes L$ of $(A \otimes C_0(G/N)) \rtimes_\gamma L$ onto $\mathrm{Ind}_G^Q \tau \rtimes_{\zeta\otimes\mathrm{lt}} L$. The imprimitivity bimodule $Z_{G/H}^{G/N}(\alpha)$ is then defined by requiring the diagram

$$\begin{array}{ccc} ((A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G) \rtimes_{\beta|} (H/N) & \xrightarrow{Z_{G/H}^{G/N}(\alpha)} & (A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\mathrm{lt}} G \\ {\scriptstyle(\Theta\rtimes L)\circ i}\downarrow\cong & & \cong\downarrow{\scriptstyle\Omega\rtimes G} \\ \mathrm{Ind}_G^Q \tau \rtimes_{\zeta\otimes\mathrm{lt}} L & \xrightarrow{W(Q)} & \mathrm{Ind}_L^Q \zeta \rtimes_{\tau\otimes\mathrm{rt}} G \end{array} \tag{4.11}$$

to commute.

4.3. Realising $K(\epsilon) \rtimes_{\epsilon_K|} (G/H)$ as a symmetric-imprimitivity bimodule

The most difficult bimodule in Theorem 4.2 is the crossed-product Katayama bimodule $K(\epsilon) \rtimes_{\epsilon_K|} (G/H)$. The difficulty arises partly because of the coaction crossed-product, and partly because the Katayama bimodule is inherently spatial. We were able to obtain this realisation by looking at a set-up which should implement $K(\epsilon)$, and then adding G/H with the appropriate group actions.

Consider the symmetric imprimitivity data

$$\begin{array}{ccccc} & & R = G/N \times G \times G/H & & \\ & \overset{(tN,s)\cdot(rN,u,vH)=(trN,su,tvH)}{\nearrow} & & \overset{(rN,u,vH)\cdot y=(ryN,uy,vH)}{\nwarrow} & \\ K = G/N \times G & & & & G \\ & \underset{\sigma=\mathrm{id}\times\alpha}{\searrow} & & \underset{\tau=\mathrm{id}}{\swarrow} & \\ & & A & & \end{array} \tag{4.12}$$

(note that K and σ are the same as for P, and τ is the same as for Q). The symmetric imprimitivity theorem gives an $(\mathrm{Ind}_G^R \tau \rtimes_{\sigma\otimes\mathrm{lt}} K)$–$(\mathrm{Ind}_K^R \sigma \rtimes_{\tau\otimes\mathrm{rt}} G)$ imprimitivity bimodule $W(R) = W({}_K R_G, A, \sigma, \tau)$.

Proposition 4.4. *Suppose $\alpha : G \to \mathrm{Aut}\, A$ is a continuous action of a locally compact group G by automorphisms of a C^*-algebra A, and suppose N and H are closed normal subgroups of G with $N \subseteq H$. Let $K(\epsilon)$ be the Katayama bimodule as defined at* (2.1) *associated to the maximal coaction $\epsilon = \hat{\alpha}|_{G/N}$ of G/N on $A \rtimes_\alpha G$, and let $W(R)$ be the bimodule associated to the symmetric imprimitivity data $({}_K R_G, A, \sigma, \tau)$ described at* (4.12)*. Then there exist* (*noncanonical*) *equivariant isomorphisms $\Lambda : A \otimes C_0(G/N) \otimes C_0(G/H) \to \mathrm{Ind}_G^R \tau$ and $\Xi : A \otimes C_0(G/H) \to \mathrm{Ind}_K^R \sigma$ such that the diagram*

$$\begin{array}{ccc} A \rtimes_\alpha G \rtimes_\epsilon (G/N) \rtimes_{\hat{\epsilon}} (G/N) \rtimes_{\hat{\epsilon}|} (G/H) & \xrightarrow{K(\epsilon)\rtimes(G/H)} & A \rtimes_\alpha G \rtimes_{\epsilon|} (G/H) \\ \cong \big\downarrow & & \big\downarrow \cong \\ (((A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G) \otimes C_0(G/H)) \rtimes_{\beta\otimes\mathrm{lt}} (G/N) & & (A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\mathrm{lt}} G \\ (\Lambda\rtimes K)\circ\iota \big\downarrow & & \big\downarrow \Xi\rtimes G \\ \mathrm{Ind}_G^R \tau \rtimes_{\sigma\otimes\mathrm{lt}} K & \xrightarrow{W(R)} & \mathrm{Ind}_K^R \sigma \rtimes_{\tau\otimes\mathrm{rt}} G \end{array} \tag{4.13}$$

of imprimitivity bimodules commutes, where the unnamed isomorphisms are the canonical ones.

Proof. The map $\Lambda : A \otimes C_0(G/N) \otimes C_0(G/H) \to \mathrm{Ind}_G^R \tau$ defined by

$$\Lambda(f)(rN, u, vH) = f\big(ur^{-1}N, vH\big) \tag{4.14}$$

is an υ–$(\sigma \otimes \mathrm{lt})$ equivariant isomorphism with inverse given by $\Lambda^{-1}(g)(rN, vH) = g(r^{-1}N, e, vH)$. So Λ induces an isomorphism $\Lambda \rtimes K$ of $((A \otimes C_0(G/N) \otimes C_0(G/H)) \rtimes_\upsilon K$ onto $\mathrm{Ind}_G^R \tau \rtimes_{\sigma\otimes\mathrm{lt}} K$. The map $\Xi : A \otimes C_0(G/H) \to \mathrm{Ind}_K^R \sigma$ defined by

$$\Xi(f)(rN, u, vH) = \alpha_u\big(f\big(r^{-1}vH\big)\big) \tag{4.15}$$

is an $(\alpha \otimes \mathrm{lt}) - (\tau \otimes \mathrm{rt})$ equivariant isomorphism with inverse given by $\Xi^{-1}(g)(wH) = g(N, e, wH)$. So Ξ also induces an isomorphism $\Xi \rtimes G$ of the crossed products.

We now define an imprimitivity bimodule W to be $W(R)$ with the coefficient algebras adjusted using these isomorphisms. Thus, the following diagram commutes by definition:

$$\begin{array}{ccc} (((A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G) \otimes C_0(G/H)) \rtimes_{\beta\otimes\mathrm{lt}} (G/N) & \xrightarrow{\;W\;} & (A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\mathrm{lt}} G \\ {\scriptstyle (\Lambda\rtimes K)\circ\iota}\Big\downarrow \cong & & \cong \Big\downarrow {\scriptstyle \Xi\rtimes G} \\ \mathrm{Ind}_G^R \tau \rtimes_{\sigma\otimes\mathrm{lt}} K & \xrightarrow{\;W(R)\;} & \mathrm{Ind}_K^R \sigma \rtimes_{\tau\otimes\mathrm{rt}} G. \end{array} \tag{4.16}$$

The formulas for the actions and inner products of W are as follows: for $b \in C_c(G/N \times G/H \times G \times G/N, A)$, $f, g \in C_c(G/N \times G \times G/H, A) \subseteq W$ and $c \in C_c(G \times G/H, A)$,

$$\begin{aligned} (b \cdot f)(rN, u, vH) &= \int_{G/N}\int_G b(tN, vH, s, rN)\alpha_s\big(f\big(s^{-1}rtN, s^{-1}u, t^{-1}vH\big)\big) \\ &\qquad \Delta_G(s)^{1/2}\Delta_{G/N}(tN)^{1/2}\,ds\,d(tN), \\ (f \cdot c)(rN, u, vH) &= \int_G f(rN, uy, vH)\alpha_{uy}\big(c\big(y^{-1}, y^{-1}u^{-1}rvH\big)\big)\Delta_G(y)^{-1/2}\,dy, \\ {}_*\langle f, g\rangle(tN, vH, s, rN) &= \Delta_G(s)^{-1/2}\Delta_{G/N}(tN)^{-1/2}\int_G f(rN, y, vH) \\ &\qquad \alpha_s\big(g\big(s^{-1}rtN, s^{-1}y, t^{-1}vH\big)^*\big)\,dy, \\ \langle f, g\rangle_*(y, wH) &= \Delta_G(y)^{-1/2}\int_G\int_{G/N}\alpha_s\big(f\big(s^{-1}tN, s^{-1}, t^{-1}wH\big)^* \\ &\qquad g\big(s^{-1}tN, s^{-1}y, t^{-1}wH\big)\big)\,d(tN)\,ds. \end{aligned} \tag{4.17}$$

To complete the proof of Proposition 4.4, it suffices to show that W and $K(\epsilon) \rtimes_{\epsilon_K|} (G/H)$ are isomorphic, modulo the canonical isomorphisms of the coefficient algebras. This will involve a spatial argument. Recall from [3, Definition 2.1] that a representation of an A–B imprimitivity bimodule X on a pair of Hilbert spaces $(\mathcal{H}_l, \mathcal{H}_r)$ is a triple (μ_l, μ, μ_r) consisting of nondegenerate representations $\mu_l : A \to B(\mathcal{H}_l)$, $\mu_r : B \to B(\mathcal{H}_r)$, and a linear map $\mu : X \to B(\mathcal{H}_r, \mathcal{H}_l)$ such that, for all $x, y \in X$, $a \in A$ and $b \in B$,

(1) $\mu(x)^*\mu(y) = \mu_r(\langle x, y\rangle_B)$ and $\mu(x)\mu(y)^* = \mu_l({}_A\langle x, y\rangle)$ and
(2) $\mu(a \cdot x \cdot b) = \mu_l(a)\mu(x)\mu_r(b)$.

The representation (μ_l, μ, μ_r) is faithful if either μ_l or μ_r is isometric (for then μ is also isometric).

Lemma 4.5. *Let (μ_l, μ, μ_r) be a faithful representation of an imprimitivity bimodule ${}_AX_B$ on a pair of Hilbert spaces $(\mathcal{H}_l, \mathcal{H}_r)$. Let ${}_{\delta_A}\delta_{\delta_B}$ be a full coaction of G on ${}_AX_B$, so that $X \rtimes_\delta G$*

is an $(A \rtimes_{\delta_A} G)$–$(B \rtimes_{\delta_B} G)$ imprimitivity bimodule. Let $\mu_{\mathrm{l}} \rtimes G$ and $\mu_{\mathrm{r}} \rtimes G$ be the regular representations of $A \rtimes_{\delta_A} G$ and $B \rtimes_{\delta_B} G$ induced from μ_{l} and μ_{r}, respectively, and let

$$\mu \rtimes G := (\mu \otimes \lambda) \circ \delta \rtimes (1 \otimes M) : X \rtimes_\delta G \to B\big(\mathcal{H}_{\mathrm{r}} \otimes L^2(G), \mathcal{H}_{\mathrm{l}} \otimes L^2(G)\big).$$

Then $(\mu_{\mathrm{l}} \rtimes G, \mu \rtimes G, \mu_{\mathrm{r}} \rtimes G)$ is a faithful representation of $X \rtimes_\delta G$.

Proof. This is essentially Theorem 3.2 of [3]. However, since it is proved there for reduced coactions, we outline an alternative proof based on results in [6]. The representations $(\mu_{\mathrm{l}}, \mu, \mu_{\mathrm{r}})$ combine to give a faithful representation $L(\mu)$ of the linking algebra $L(X)$ as bounded operators on $\mathcal{H}_{\mathrm{l}} \oplus \mathcal{H}_{\mathrm{r}}$. As in [6, Chapter 3, §1.2], the coactions μ_{l}, μ, and μ_{r} combine to give a coaction ν of G on $L(X)$, and $L(X) \rtimes_\nu G$ is canonically isomorphic to $L(X \rtimes_\delta G)$ [6, Lemma 3.10]. The regular representation $L(\mu) \rtimes G$ of $L(X) \rtimes_\nu G$ on

$$(\mathcal{H}_{\mathrm{l}} \oplus \mathcal{H}_{\mathrm{r}}) \otimes L^2(G) = \big(\mathcal{H}_{\mathrm{l}} \otimes L^2(G)\big) \oplus \big(\mathcal{H}_{\mathrm{r}} \otimes L^2(G)\big)$$

is faithful by [6, Corollary A.59]. Since

$$L(\mu) \rtimes G = \big(\big(L(\mu) \otimes \lambda\big) \circ \nu\big) \rtimes (1 \otimes M)$$

restricts to the regular representations on the corners of $L(X \rtimes_\delta G)$, we deduce that $\mu \rtimes G$ is faithful too. □

Conclusion of the proof of Proposition 4.4. Let (π, U) be a faithful covariant representation of (A, G, α) on a Hilbert space $\mathcal{H}$. The idea of the proof is to find faithful representations $(\nu_{\mathrm{l}}, \nu, \nu_{\mathrm{r}})$ and $(\mu_{\mathrm{l}} \rtimes (G/H), \mu \rtimes (G/H), \mu_{\mathrm{r}} \rtimes (G/H))$ of W and $K(\epsilon) \rtimes_{\epsilon_{K|}} (G/H)$ on

$$\big(\mathcal{H} \otimes L^2(G/N) \otimes L^2(G/H), \mathcal{H} \otimes L^2(G/H)\big)$$

such that the ranges of ν_{r} and $\mu_{\mathrm{r}} \rtimes (G/H)$ coincide. We will then argue that a dense subset of the range of ν is contained in the range of $\mu \rtimes G/H$. Thus W is isomorphic to a closed submodule of $K(\epsilon) \rtimes_{\epsilon_{K|}} (G/H)$ on which the right inner product is full, and it then follows from the Rieffel correspondence that W and $K(\epsilon) \rtimes_{\epsilon_{K|}} (G/H)$ are isomorphic.

The representation

$$\nu_{\mathrm{r}} := \big(\pi \otimes M^{G/H}\big) \rtimes \big(U \otimes \lambda^{G/H}\big) : \big(A \otimes C_0(G/H)\big) \rtimes_{\alpha \otimes \mathrm{lt}} G \to B\big(\mathcal{H} \otimes L^2(G/H)\big)$$

is faithful; for future use, note that it is given on the pieces A, $C^*(G)$ and $C_0(G/H)$ by $\pi \otimes 1$, $U \otimes \lambda^{G/H}$ and $1 \otimes M^{G/H}$, respectively. Let

$$\nu_{\mathrm{l}} : \big(\big(\big(A \otimes C_0(G/N)\big) \rtimes_{\alpha \otimes \mathrm{lt}} G\big) \otimes C_0(G/H)\big) \rtimes_{\beta \otimes \mathrm{lt}} G/N \to B\big(\mathcal{H} \otimes L^2(G/N) \otimes L^2(G/H)\big)$$

be the representation

$$\nu_{\mathrm{l}} := \big(\big(\pi \otimes M^{G/N} \rtimes U \otimes \lambda^{G/N}\big) \otimes M^{G/N}\big) \rtimes 1 \otimes \rho \otimes \lambda^{G/H};$$

it is given on the pieces A, $C^*(G)$, $C_0(G/N)$, $C^*(G/N)$ and $C_0(G/H)$) by $\pi \otimes 1 \otimes 1$, $U \otimes \lambda^{G/N} \otimes 1$, $1 \otimes M^{G/N} \otimes 1$, $1 \otimes \rho \otimes \lambda^{G/H}$ and $1 \otimes 1 \otimes M^{G/H}$, respectively. Next, we claim that, for fixed $z \in C_c(G/N \times G \times G/H, A) \subseteq W$ and every $\xi \in L^2(G/H, \mathcal{H})$, the map $\nu(z)\xi$ of $G/N \times G/H$ into $\mathcal{H}$ given by

$$\big(\nu(z)\xi\big)(rN, vH) = \int_G \pi\big(z(rN, y, vH)\big)U_y\xi\big(y^{-1}rvH\big)\Delta_G(y)^{-1/2}\,dy \tag{4.18}$$

is an element of $L^2(G/N \times G/H, \mathcal{H}) \cong \mathcal{H} \otimes L^2(G/N) \otimes L^2(G/H)$. For $w \in C_c(G/N \times G \times G/H, A)$ and $\eta \in L^2(G/H, \mathcal{H})$, we have

$$\begin{aligned}
&\int_{G/N\times G/H} \big(\nu(w)\eta(rN, vH) \,\big|\, \nu(z)\xi(rN, vH)\big)\,d(rN, vH) \\
&= \int_{G/N}\int_{G/H}\int_G\int_G \big(\pi\big(w(rN, x, vH)\big)U_x\eta\big(x^{-1}rvH\big) \,\big|\, \pi\big(z(rN, y, vH)\big)U_y\xi\big(y^{-1}rvH\big)\big) \\
&\quad \Delta_G(xy)^{-1/2}\,dx\,dy\,d(vH)\,d(rN) \\
&\overset{(\dagger)}{=} \int_{G/N}\int_{G/H}\int_G\int_G \big(\pi\big(w\big(y^{-1}rN, y^{-1}x, r^{-1}vH\big)\big)U_{y^{-1}x}\eta\big(x^{-1}vH\big) \\
&\quad \big|\, \pi\big(z\big(y^{-1}rN, y^{-1}, r^{-1}vH\big)\big)U_{y^{-1}}\xi(vH)\big)\Delta_G(x)^{-1/2}\,dx\,dy\,d(vH)\,d(rN) \\
&= \int_{G/H}\int_G\int_G\int_{G/N} \big(U_y\pi\big(z\big(y^{-1}rN, y^{-1}, r^{-1}vH\big)^* w\big(y^{-1}rN, y^{-1}x, r^{-1}vH\big)\big)U_{y^{-1}x}\eta\big(x^{-1}vH\big) \\
&\quad \big|\, \xi(vH)\big)\Delta_G(x)^{-1/2}\,d(y, rN)\,dx\,d(vH) \\
&= \int_{G/H}\int_G \bigg(\pi\bigg(\int_G\int_{G/N} \alpha_y\big(z\big(y^{-1}rN, y^{-1}, r^{-1}vH\big)^* w\big(y^{-1}rN, y^{-1}x, r^{-1}vH\big)\big)\,d(y)\,d(rN)\bigg) \\
&\quad U_x\eta\big(x^{-1}vH\big) \,\big|\, \xi(vH)\bigg)\Delta_G(x)^{-1/2}\,dx\,d(vH) \\
&= \int_{G/H}\int_G \big(\pi\big(\langle z, w\rangle_*(x, vH)\big)U_x\eta\big(x^{-1}vH\big) \,\big|\, \xi(vH)\big)\,dx\,d(vH) \\
&= \int_{G/H} \big(\big(\big(\pi \otimes M^{G/H}\big) \rtimes \big(U \otimes \lambda^{G/H}\big)\big(\langle z, w\rangle_*\big)\eta\big)(vH) \,\big|\, \xi(vH)\big)\,d(vH) \\
&= \big(\nu_{\mathrm{r}}\big(\langle z, w\rangle_*\big)\eta \,\big|\, \xi\big),
\end{aligned}$$

where $(\cdot \mid \cdot)$ denotes the appropriate Hilbert space inner product. The change of variables at $(\dagger)$ is given by $(vH, rN, x, y) \mapsto (r^{-1}vH, y^{-1}rN, y^{-1}x, y^{-1})$. In particular, this shows that $\|\nu(z)\xi\|_2^2 = (\nu(z)\xi \mid \nu(z)\xi) \leqslant \|z\|^2\|\xi\|^2$, so $\nu(z)\xi \in L^2(G/N \times G/H, \mathcal{H})$, and that the linear map $\xi \mapsto \nu(z)\xi$ is bounded. Thus ν, as defined at (4.18), extends to a linear map

$$\nu : W \to B\big(\mathcal{H} \otimes L^2(G/H), \mathcal{H} \otimes L^2(G/N) \otimes L^2(G/H)\big).$$

We claim that $(\nu_{\mathrm{l}}, \nu, \nu_{\mathrm{r}})$ is a representation of W. We will prove that, for $z, w \in C_{\mathrm{c}}(G/N \times G \times G/H, A) \subseteq W$ and $b \in C_{\mathrm{c}}(G/N \times G/H \times G \times G/N, A)$,

(1) $\nu(z)^*\nu(w) = \nu_{\mathrm{r}}(\langle z, w\rangle_*)$ in $B(\mathcal{H} \otimes L^2(G/H))$;
(2) $\nu(b \cdot z) = \nu_{\mathrm{l}}(b)\nu(z)$; and
(3) ν is nondegenerate in the sense that $\{\nu(z)\xi\colon\ z \in W,\ \xi \in \mathcal{H} \otimes L^2(G/H)\}$ is dense in $\mathcal{H} \otimes L^2(G/N) \otimes L^2(G/H)$.

Then (1) implies that $\nu(z \cdot c) = \nu(z)\nu_{\mathrm{r}}(c)$ for all $c \in (A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\mathrm{lt}} G$, and the other inner product condition follows from this and (1)–(3).

To see that (1) holds, it suffices to see that

$$\big(\nu(w)\eta \bigm| \nu(z)\xi\big) = \big(\nu_{\mathrm{r}}\big(\langle z, w\rangle_*\big)\eta \bigm| \xi\big)$$

for all $z, w \in C_{\mathrm{c}}(G/N \times G \times G/H, A)$ and $\xi, \eta \in L^2(G/H, \mathcal{H})$, and this was done in the calculation above which showed ν is well defined.

It will be easiest to check (2) on the separate pieces of the algebra. The piece $A \otimes C_0(G/N) \otimes C_0(G/H)$ is represented by $\pi \otimes M^{G/N} \otimes M^{G/H}$ and we deduce from (4.17) that $b \in C_{\mathrm{c}}(G/N \times G/H, A)$ acts on $z \in C_{\mathrm{c}}(G/N \times G \times G/H, A) \subseteq W$ by

$$(b \cdot z)(rN, u, vH) = b(rN, vH)z(rN, u, vH).$$

The group $G \times G/N$ is represented by $(U \otimes \lambda^{G/N} \otimes 1) \times (1 \otimes \rho \otimes \lambda^{G/H})$ and acts on W by

$$\big((s, tN) \cdot z\big)(rN, u, vH) = \alpha_s\big(z\big(s^{-1}rtN, s^{-1}u, t^{-1}vH\big)\big)\Delta_G(s)^{1/2}\Delta_{G/N}(tN)^{1/2}.$$

Thus, for $\xi \in L^2(G/H, \mathcal{H})$,

$$\begin{aligned}
\big(\nu(b \cdot z)\xi\big)(rN, vH) &= \int_G \pi\big((b \cdot z)(rN, y, vH)\big)U_y\xi\big(y^{-1}rvH\big)\Delta_G(y)^{-1/2}\,dy \\
&= \int_G \pi\big(b(rN, vH)z(rN, y, vH)\big)U_y\xi\big(y^{-1}rvH\big)\Delta_G(y)^{-1/2}\,dy \\
&= \pi\big(b(rN, vH)\big)\big(\big(\nu(z)\xi\big)(rN, vH)\big) \\
&= \big(\big(\pi \otimes M^{G/N} \otimes M^{G/H}\big)(b)\big)\big(\nu(z)\xi\big)(rN, vH) \\
&= \nu_{\mathrm{l}}(b)\big(\nu(z)\xi\big)(rN, vH),
\end{aligned}$$

and

$$\begin{aligned}
&\big(\nu\big((s, tN) \cdot z\big)\xi\big)(rN, vH) \\
&\quad = \int_G \pi\big(\big((s, tN) \cdot z\big)(rN, y, vH)\big)U_y\xi\big(y^{-1}rvH\big)\Delta_G(y)^{-1/2}\,dy
\end{aligned}$$

$$= \int_G \pi\bigl(\alpha_s\bigl(z\bigl(s^{-1}rtN, s^{-1}y, t^{-1}vH\bigr)\bigr)\bigr)U_y\xi\bigl(y^{-1}rvH\bigr)\Delta_G(s)^{1/2}\Delta_{G/N}(tN)^{1/2}\Delta_G(y)^{-1/2}\,dy$$

$$= \int_G U_s\pi\bigl(z\bigl(s^{-1}rtN, s^{-1}y, t^{-1}vH\bigr)\bigr)U_{s^{-1}y}\xi\bigl(y^{-1}rvH\bigr)\Delta_G(s)^{1/2}\Delta_{G/N}(tN)^{1/2}\Delta_G(y)^{-1/2}\,dy$$

$$\overset{(\dagger)}{=} U_s\int_G \pi\bigl(z\bigl(s^{-1}rtN, y, t^{-1}vH\bigr)\bigr)U_y\xi\bigl(y^{-1}s^{-1}rvH\bigr)\Delta_{G/N}(tN)^{1/2}\Delta_G(y)^{-1/2}\,dy$$

$$= U_s\bigl(\nu(z)\xi\bigr)\bigl(s^{-1}rtN, t^{-1}vH\bigr)\Delta_{G/N}(tN)^{1/2}$$

$$= \bigl((U_s\otimes\lambda_s\otimes 1)\bigl(1\otimes\rho_{tN}\otimes\lambda_{tN}^{G/H}\bigr)\bigr)\bigl(\nu(z)\xi\bigr)(rN, vH)$$

$$= \nu_1\bigl((s,tN)\bigr)\bigl(\nu(z)\xi\bigr)(rN, vH),$$

where the change of variables at (†) was $y\mapsto sy$. Thus (2) holds.

For (3), fix $\zeta>0$. Also fix nonzero $\varphi\in C_c(G/N)$, $\eta\in C_c(G/H)$ and $h\in\mathcal{H}$. It suffices to approximate (in $L^2(G/N\times G/H,\mathcal{H})$) the function

$$(rN, vH)\mapsto\varphi(rN)\eta(vH)h.$$

Using an approximate identity and the nondegeneracy of π, choose nonzero $a\in A$ such that $\|\pi(a)h-h\|<\zeta/(2\|\varphi\otimes\eta\|_2)$. Then choose a relatively compact open neighbourhood O of e in G such that $\|U_yh-h\|<\zeta/(2\|a\|\|\varphi\otimes\eta\|_2)$ for $y\in O$. Then

$$\bigl\|\pi(a)U_yh-h\bigr\|<\frac{\zeta}{\|\varphi\otimes\eta\|_2}$$

for all $y\in O$. Next, choose $f\in C_c(G)$ with $\operatorname{supp} f\subseteq O$ such that $\int_G f(y)\Delta_G(y)^{-1/2}\,dy=1$. Also choose compact subsets L and K of G such that $L/N=\operatorname{supp}\varphi$ and $K/H=\operatorname{supp}\eta$, and $\xi\in C_c(G/H,\mathcal{H})$ such that $\xi(s)=h$ for $sH\in O^{-1}LK/H$. Set

$$z(rN, y, vH)=\varphi(rN)f(y)\eta(vH)a.$$

Then $z\in C_c(G/N\times G\times G/H, A)$ and

$$\bigl\|\nu(z)\xi-\varphi\otimes\eta\otimes h\bigr\|_2^2$$

$$=\int_{G/N}\int_{G/H}\bigl\|\nu(z)\xi(rN,vH)-\varphi(rN)\eta(vH)h\bigr\|^2\,d(rN)\,d(vH)$$

$$=\int_{G/N}\int_{G/H}\biggl\|\int_G\varphi(rN)\eta(vH)f(y)\pi(a)U_y\xi\bigl(y^{-1}rvH\bigr)\Delta_G(y)^{-1/2}\,dy$$

$$-\biggl(\int_G f(y)\Delta_G(y)^{-1/2}\,dy\biggr)\varphi(rN)\eta(vH)h\biggr\|^2\,d(rN)\,d(vH)$$

$$\leqslant \int_{G/N}\int_{G/H} \|\varphi(rN)\eta(vH)\|^2\biggl(\int_G \|f(y)\Delta_G(y)^{-1/2}\|\|\pi(a)U_y h - h\|\,dy\biggr)^2\,d(rN)\,d(vH)$$

by our choice of ξ. Since $\|\pi(a)U_y h - h\| < \zeta/(\|\varphi\otimes\eta\|_2)$ for all $y \in \operatorname{supp} f$ we have

$$\|\nu(z)\xi - \varphi\otimes\eta\otimes h\|_2 < \zeta,$$

and hence ν is nondegenerate. Thus $(\nu_{\mathrm{l}}, \nu, \nu_{\mathrm{r}})$ is a representation of W; the faithfulness follows because ν_{r} is faithful.[1]

We will obtain our representation of $K(\epsilon) \rtimes_{\epsilon_K|} (G/H)$ by first constructing a representation $(\mu_{\mathrm{l}}, \mu, \mu_{\mathrm{r}})$ of $K(\epsilon) = (A \rtimes_\alpha G) \otimes L^2(G/N)$ on the pair $(\mathcal{H}\otimes L^2(G/N), \mathcal{H})$ of Hilbert spaces, and then applying Lemma 4.5 to the coaction $\epsilon_K|$ of G/H on $K(\epsilon)$.

We represent $A \rtimes_\alpha G \rtimes_\epsilon (G/N) \rtimes_{\hat\epsilon} (G/N)$ on $\mathcal{H}\otimes L^2(G/N)$ and $A\rtimes_\alpha G$ on $\mathcal{H}$ by

$$\begin{aligned}\mu_{\mathrm{l}} &:= \bigl(\bigl((\pi \rtimes U)\otimes\lambda\bigr)\circ\epsilon\bigr) \rtimes \bigl(1\otimes M^{G/N}\bigr) \rtimes (1\otimes\rho)\\ &= (\pi\otimes 1) \rtimes \bigl(U\otimes\lambda^{G/N}\bigr) \rtimes \bigl(1\otimes M^{G/N}\bigr) \rtimes (1\otimes\rho)\end{aligned}$$

and $\mu_{\mathrm{r}} := \pi \rtimes U$, respectively, and let $\mu : K(\epsilon) \to B(\mathcal{H}, \mathcal{H}\otimes L^2(G/N))$ be the linear map such that

$$\mu(b\otimes f)h = \pi \rtimes U(b)h\otimes f \quad \bigl(b \in A\rtimes_\alpha G,\ f\in L^2(G/H),\ h\in\mathcal{H}\bigr).$$

Note that μ is nondegenerate because $\pi\rtimes U$ is. For $b\otimes f, c\otimes g \in K(\epsilon)$ and $h,k\in\mathcal{H}$,

$$\begin{aligned}\bigl(\mu(b\otimes f)h \bigm| \mu(c\otimes g)k\bigr) &= \bigl(\pi\rtimes U(b)h\otimes f \bigm| \pi\rtimes U(c)k\otimes g\bigr)\\ &= \bigl(\pi\rtimes U\bigl(c^*b\bigr)h \bigm| k\bigr)(g\mid f)\\ &= \bigl(\pi\rtimes U\bigl(c^*b(g\mid f)\bigr)h \bigm| k\bigr)\\ &= \bigl(\pi\rtimes U\bigl(\langle c\otimes g, b\otimes f\rangle_{A\rtimes_\alpha G}\bigr)h \bigm| k\bigr)\\ &= \bigl(\mu_{\mathrm{r}}\bigl(\langle c\otimes g, b\otimes f\rangle_{A\rtimes_\alpha G}\bigr)h \bigm| k\bigr),\end{aligned}$$

so $\mu(b\otimes f)^*\mu(b\otimes f) = \mu_{\mathrm{r}}(\langle b\otimes f, b\otimes f\rangle_{A\rtimes_\alpha G})$.

On the pieces $A, C^*(G), C_0(G/N)$ and $C^*(G/N)$, μ_{l} is given by $\pi\otimes 1$, $U\otimes\lambda^{G/N}$, $1\otimes M^{G/N}$ and $1\otimes\rho$, respectively. Let (i_A, i_G) be the universal covariant representation of (A,G,α). The left action of $A\rtimes_\alpha G\rtimes_\epsilon (G/N)\rtimes_{\hat\epsilon}(G/N)$ on $K(\epsilon)$ is via the isomorphism

$$\bigl((\mathrm{id}\otimes\lambda)\circ\epsilon\bigr)\rtimes\bigl(1\otimes M^{G/N}\bigr)\rtimes(1\otimes\rho) = (i_A\otimes 1)\rtimes\bigl(i_G\otimes\lambda^{G/N}\bigr)\rtimes\bigl(1\otimes M^{G/N}\bigr)\rtimes(1\otimes\rho)$$

of $A\rtimes_\alpha G\rtimes_\epsilon (G/N)\rtimes_{\hat\epsilon}(G/N)$ onto $(A\rtimes_\alpha G)\otimes\mathcal{K}(L^2(G/N))$. For $a\in A$, $g\in C_0(G/N)$ and $s\in G$ we have

[1] Note that the proof that $(\nu_{\mathrm{l}}, \nu, \nu_{\mathrm{r}})$ is a faithful representation did not require H to be normal in G, but we do need the normality in what follows.

$$\begin{aligned}
\mu_{\mathrm{l}}(a)\mu(b\otimes f)h &= \big(\pi(a)\otimes 1\big)\big(\pi\rtimes U(b)h\otimes f\big)\\
&= \pi\rtimes U\big(i_A(a)b\big)h\otimes f\\
&= \mu\big(\big(i_A(a)\otimes 1\big)(b\otimes f)\big)h;\\
\mu_{\mathrm{l}}(s)\mu(b\otimes f)h &= \big(U_s\otimes \lambda_s^{G/N}\big)\big(\pi\rtimes U(b)h\otimes f\big)\\
&= \pi\rtimes U\big(i_G(s)b\big)h\otimes \lambda_s^{G/N}(f)\\
&= \mu\big(\big(i_G(s)\otimes \lambda_s^{G/N}\big)(b\otimes f)\big)h;\\
\mu_{\mathrm{l}}(g)\mu(b\otimes f)h &= \big(1\otimes M^{G/N}(g)\big)\big(\pi\rtimes U(b)h\otimes f\big)\\
&= \pi\rtimes U(b)h\otimes M^{G/N}(g)f\\
&= \mu\big(\big(1\otimes M^{G/N}(g)\big)(b\otimes f)\big)h;\\
\mu_{\mathrm{l}}(sN)\mu(b\otimes f)h &= 1\otimes \rho_{sN}\big(\pi\rtimes U(b)h\otimes f\big)\\
&= \pi\rtimes U(b)h\otimes \rho_{sN}(f)\\
&= \mu\big(1\otimes \rho_{sN}(b\otimes f)\big)h.
\end{aligned}$$

It follows that $(\mu_{\mathrm{l}},\mu,\mu_{\mathrm{r}})$ is a representation of $K(\epsilon)$ on $(\mathcal{H}\otimes L^2(G/N),\mathcal{H})$.

The coaction ϵ_K of G/N on $K(\epsilon)$ is defined in [13, Proposition 4.2] by

$$\epsilon_K(b\otimes f) = V\big(\epsilon(b)\otimes f\big)^{\Sigma_{23}}$$

(here $\Sigma_{23}(b\otimes f\otimes g) = b\otimes g\otimes f$ and $V\in M((A\rtimes_\alpha G)\otimes\mathcal{K}(L^2(G/N))\otimes C^*(G/N))$ is given by

$$V = 1\otimes\big(M^{G/N}\otimes \mathrm{id}\big)\big(w_{G/N}^*\big),$$

where $w_{G/N}\in UM(C_0(G/N)\otimes C^*(G/N))$ is the usual multiplicative unitary). By Lemma 4.5, $(\mu_{\mathrm{l}}\rtimes(G/H),\mu\rtimes(G/H),\mu_{\mathrm{r}}\rtimes(G/H))$ is a faithful representation of $K\rtimes_{\epsilon|}(G/H)$ on the subspace

$$\overline{\operatorname{span}}\big\{\big((\mu\otimes\lambda)\circ\epsilon_K\big|(b\otimes f)\big)(1\otimes M_g)\;\big|\; b\in A\rtimes_\alpha G,\ f\in L^2(G/N),\ g\in C_0(G/H)\big\} \tag{4.19}$$

of $B(\mathcal{H}\otimes L^2(G/H),\mathcal{H}\otimes L^2(G/N)\otimes L^2(G/H))$. Since the canonical isomorphism of $(A\otimes C_0(G/H))\rtimes_{\alpha\otimes\mathrm{lt}} G$ onto $A\rtimes_\alpha G\rtimes_\epsilon (G/H)$ carries the representation $\nu_{\mathrm{r}} = (\pi\otimes M^{G/H})\rtimes(U\otimes\lambda^{G/H})$ into $\mu_{\mathrm{r}}\rtimes(G/H) = (((\pi\rtimes U)\otimes\lambda^{G/H})\circ\epsilon)\rtimes(1\otimes M^{G/H})$, the ranges of ν_{r} and $\mu_{\mathrm{r}}\rtimes(G/H)$ clearly coincide. (It is also not hard to check that the canonical isomorphism of the left-hand coefficient algebras carries ν_{l} into $\mu_{\mathrm{l}}\rtimes(G/H)$.)

To finish the proof of the theorem, we need to show that the ranges of ν and $\mu\rtimes(G/H)$ coincide. By the Rieffel correspondence it suffices to show that a dense subset of the range of ν is contained in the range of $\mu\rtimes(G/H)$. To do this, we need a more useful expression for terms of the form $(\mu\otimes\lambda)\circ\epsilon_K|(b\otimes f)$. We have

$$(\mu\otimes\lambda)\circ\epsilon_K| = (\mu\otimes\lambda\circ q)\circ\epsilon_K.$$

Recall that $\epsilon = \hat{\alpha}| = (\mathrm{id} \otimes q) \circ ((i_A \otimes 1) \rtimes (i_G \otimes \iota)) = (i_A \otimes 1) \rtimes (i_G \otimes q)$, where $\iota : G \to C^*(G)$ is the canonical map, and q maps $C^*(G)$ into $C^*(G/N)$. Thus, for $b \in C_c(G, A) \subseteq A \rtimes_\alpha G$, we have

$$\epsilon(b) = \int_G (i_A \otimes 1)\big(b(s)\big)(i_G \otimes q)(s)\,ds = \int_G i_A\big(b(s)\big)i_G(s) \otimes q(s)\,ds.$$

Let $f \in L^2(G/N)$ and $h \in \mathcal{H}$. We write $c_f(h) = h \otimes f$ for $h \in \mathcal{H}$; note that

$$\mu(b \otimes f) = c_f \circ \big(\pi \rtimes U(b)\big).$$

We have

$$\big(\epsilon(b) \otimes f\big)^{\Sigma_{23}} = \int_G i_A\big(b(s)\big)i_G(s) \otimes f \otimes q(s)\,ds,$$

and therefore

$$(\mu \otimes \lambda \circ q)\big(\epsilon(b) \otimes f\big)^{\Sigma_{23}} = \int_G \big(c_f \circ \pi\big(b(s)\big)U_s\big) \otimes \lambda_{sH}\,ds.$$

In other words, for $\xi \in L^2(G/H, \mathcal{H})$,

$$(\mu \otimes \lambda \circ q)\big(\epsilon(b) \otimes f\big)^{\Sigma_{23}}\xi(rN, tH) = \int_G f(rN)\pi\big(b(s)\big)U_s\xi\big(s^{-1}tH\big)\,ds.$$

Next, for $T \in \mathcal{K}(L^2(G/N))$, $z, w \in C^*(G/N)$, $f \in L^2(G/N)$, and $b \in A \rtimes_\alpha G$, compute:

$$\begin{aligned}
&(\mu \otimes \lambda \circ q)\big((1 \otimes T \otimes z)(b \otimes f \otimes w)\big)\\
&\quad = (\mu \otimes \lambda \circ q)(b \otimes Tf \otimes zw) = c_{Tf} \circ \big(\pi \rtimes U(b)\big) \otimes \lambda \circ q(zw)\\
&\quad = \big((\mathrm{id} \otimes \mathrm{id} \otimes \lambda \circ q)(1 \otimes T \otimes z)\big)\big((\mu \otimes \lambda \circ q)(b \otimes f \otimes w)\big).
\end{aligned}$$

From this we deduce that, for multipliers of $(A \rtimes_\alpha G) \otimes \mathcal{K} \otimes C^*(G/N)$ of the form $1 \otimes m$, we have

$$(\mu \otimes \lambda \circ q)(1 \otimes m) = 1 \otimes (\mathrm{id} \otimes \lambda \circ q)(m),$$

and in particular

$$(\mu \otimes \lambda \circ q)(V) = 1 \otimes \big(M^{G/N} \otimes \lambda \circ q\big)\big(w_{G/N}^*\big);$$

that is, for $\eta \in L^2(G/N \times G/H, \mathcal{H})$,

$$(\mu \otimes \lambda \circ q)(V)\eta(rN, vH) = \eta(rN, rvH).$$

Combining the above, we get

$$\begin{aligned}(\mu\otimes\lambda)\circ\epsilon_K|(b\otimes f)\xi(rN,vH) &= (\mu\otimes\lambda\circ q)\bigl(\epsilon(b)\otimes f\bigr)^{\Sigma_{23}}\xi(rN,rvH)\\ &= \int_G f(rN)\pi\bigl(b(y)\bigr)U_y\xi\bigl(y^{-1}rvH\bigr)\,dy\end{aligned}$$

for $b\in C_c(G,A)\subseteq A\rtimes_\alpha G$, $f\in L^2(G/N)$, and $\xi\in L^2(G/H,\mathcal{H})$. The image of $K(\epsilon)\rtimes_{\epsilon_K|}(G/H)$ is thus densely spanned by the operators defined by

$$\begin{aligned}&(\mu\otimes\lambda)\circ\epsilon_K|(b\otimes f)\bigl((1\otimes M_g)\xi\bigr)(rN,vH)\\ &\quad=\int_G f(rN)\pi\bigl(b(y)\bigr)g\bigl(y^{-1}rvH\bigr)U_y\xi\bigl(y^{-1}rvH\bigr)\,dy.\end{aligned}$$

Let $z\in C_c(G/N\times G\times G/H,A)$ be the function

$$z(rN,v,vH)=f(rN)b(y)g\bigl(y^{-1}rvH\bigr)\Delta_G(y)^{1/2};$$

then

$$\begin{aligned}\bigl((\mu\rtimes G/H)(b\otimes f)(1\otimes M_g)\bigr)\xi(rN,vH) &= (\mu\otimes\lambda)\circ\epsilon_K|(b\otimes f)(1\otimes M_g)\xi(rN,vH)\\ &= \int_G \pi\bigl(z(rN,y,vH)\bigr)U_y\xi\bigl(y^{-1}rvH\bigr)\Delta_G(y)^{-1/2}\,dy\\ &= \bigl(\nu(z)\xi\bigr)(rN,vH).\end{aligned}$$

It follows that the ranges of ν and $\mu\rtimes(G/H)$ in $B(\mathcal{H}\otimes L^2(G/H),\mathcal{H}\otimes L^2(G/N)\otimes L^2(G/H))$ coincide, and this completes the proof of Proposition 4.4. □

5. Proof of Theorem 4.2

Recall that in Theorem 4.2 we assume that α is a continuous action of a locally compact group G by automorphisms of a C^*-algebra A, N and H are closed normal subgroups of G with $N\subseteq H$, and we have let ϵ denote the maximal coaction $\hat{\alpha}|_{G/N}$ of G/N on $A\rtimes_\alpha G$. We also retain the symmetric-imprimitivity bimodules $W(P)$, $W(Q)$, and $W(R)$ defined in Section 4, and all the associated notation.

The basic idea is to invoke the symmetric imprimitivity calculus of Theorem 3.1 and then show that $P\#Q$ is equivariantly isomorphic to R, so that

$$X_{H/N}^{G/N}(\hat{\epsilon})\otimes_* Z_{G/H}^{G/N}(\alpha)\cong W(P)\otimes_\Phi W(Q)\cong W(P\#Q)\cong W(R)\cong K(\epsilon)\rtimes_{\epsilon_K|}(G/H).$$

However, there are many isomorphisms of the coefficient algebras involved here (see diagram (5.8)), several of them noncanonical, and we must make sure they are all compatible with this argument.

5.1. Applying Theorem 3.1 to P and Q

The map $(rN, u, vN) \mapsto (v^{-1}rN, e)G = (v^{-1}N, r^{-1})G$ of P into Q/G induces a homeomorphism $\varphi : K\backslash P \to Q/G$ such that

$$\varphi\big(K(rN, u, vN)\big) = \big(v^{-1}N, r^{-1}\big)G.$$

Moreover, φ is L-equivariant: for $(hN, y) \in L = N/H \times G$,

$$\begin{aligned}\varphi\big(K(rN, u, vN)\cdot(hN, y)\big) &= \varphi\big(K(ryN, uy, vhN)\big)\\ &= \big(h^{-1}v^{-1}N, y^{-1}r^{-1}\big)G\\ &= \big(h^{-1}N, y^{-1}\big)\cdot\big(v^{-1}N, r^{-1}\big)G\\ &= (hN, y)^{-1}\cdot\varphi\big(K(rN, u, vN)\big).\end{aligned}$$

Thus the fibred product of P and Q over φ is

$$\begin{aligned}&P \times_\varphi Q\\ &\quad= \big\{(p, q) \in P \times Q \;\big|\; \varphi(Kp) = qG\big\}\\ &\quad= \big\{(rN, u, vN, wN, z) \in G/N \times G \times G/N \times G/N \times G \;\big|\; \big(v^{-1}N, r^{-1}\big)G = (wN, z)G\big\}\\ &\quad= \big\{(rN, u, vN, wN, z) \in G/N \times G \times G/N \times G/N \times G \;\big|\; wN = v^{-1}rzN\big\},\end{aligned}$$

and the right action of L on $P \times_\varphi Q$ is given by

$$(rN, u, vN, wN, z)\cdot(hN, y) = \big(ryN, uy, vhN, h^{-1}wN, y^{-1}z\big).$$

Now define $\tilde\sigma : P \to \operatorname{Aut} A$ and $\tilde\tau : Q \to \operatorname{Aut} A$ by

$$\tilde\sigma_{(rN,u,vN)} = \alpha_u \quad\text{and}\quad \tilde\tau_{(wN,z)} = \mathrm{id}.$$

We have

$$\tilde\sigma_{(tN,s)\cdot(rN,u,vN)\cdot(hN,y)} = \tilde\sigma_{(tryN,suy,tvhN)} = \alpha_{suy} = \alpha_s\alpha_u\alpha_y = \sigma_{(tN,s)}\tilde\sigma_{(rN,u,vN)}\zeta_{(hN,y)}$$

and

$$\tilde\tau_{(hN,s)\cdot(wN,z)\cdot y} = \tilde\tau_{(hwyN,szy)} = \mathrm{id} = \mathrm{id}\,\mathrm{id}\,\mathrm{id} = \eta_{(hN,s)}\tilde\tau_{(wN,z)}\tau_y.$$

It is clear that ζ, σ, and $\tilde\sigma$ commute with η, $\tilde\tau$, and τ, since the latter are trivial. Thus all the hypotheses of Theorem 3.1 are satisfied. Therefore there exist isomorphisms

$$\begin{aligned}\Phi &: \operatorname{Ind}_K^P \sigma \rtimes_{\eta\otimes\mathrm{rt}} L \to \operatorname{Ind}_G^Q \tau \rtimes_{\zeta\otimes\mathrm{lt}} L,\\ \Phi_\sigma &: \operatorname{Ind}_K^{P\#Q} \sigma \rtimes_{\tau\otimes\mathrm{rt}} G \to \operatorname{Ind}_L^Q \zeta \rtimes_{\tau\otimes\mathrm{rt}} G, \quad\text{and}\\ \Phi_\tau &: \operatorname{Ind}_G^{P\#Q} \rtimes_{\sigma\otimes\mathrm{lt}} K \to \operatorname{Ind}_L^P \eta \rtimes_{\sigma\otimes\mathrm{lt}} K\end{aligned}$$

such that the upper square of the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Ind}_G^{P\#Q}\tau \rtimes_{\sigma\otimes\mathrm{lt}} K & \xrightarrow{W(P\#Q)} & \mathrm{Ind}_K^{P\#Q}\sigma \rtimes_{\tau\otimes\mathrm{rt}} G \\
\Phi_\tau \Big\downarrow \cong & & \Phi_\sigma \Big\downarrow \cong \\
\mathrm{Ind}_L^P \eta \rtimes_{\sigma\otimes\mathrm{lt}} K & \xrightarrow{W(P)\otimes_\Phi W(Q)} & \mathrm{Ind}_L^Q\zeta \rtimes_{\tau\otimes\mathrm{rt}} G \\
W(P)\Big\downarrow & & \Big\uparrow W(Q)\\
\mathrm{Ind}_K^P\sigma \rtimes_{\eta\otimes\mathrm{rt}} L & \xrightarrow[\cong]{\Phi} & \mathrm{Ind}_G^Q \tau \rtimes_{\zeta\otimes\mathrm{lt}} L.
\end{array} \tag{5.1}$$

The lower square commutes by definition of $W(P)\otimes_\Phi W(Q)$.

Since we will need it later, we recall from Lemma 3.2 that the isomorphism Φ is induced by the L-equivariant isomorphism $T:\mathrm{Ind}_K^P\sigma \to \mathrm{Ind}_G^Q\tau$ defined by

$$T(f)(rN,s) = f\big(s^{-1}N, e, r^{-1}N\big) \tag{5.2}$$

(because $\varphi(K(s^{-1}N,e,r^{-1}N)) = (rs^{-1}N,e)G = (rN,s)G$). Further, the isomorphism Φ_σ is induced by the G-equivariant isomorphism $\varphi_\sigma:\mathrm{Ind}_K^{P\#Q}\sigma \to \mathrm{Ind}_L^Q\zeta$ given by

$$\varphi_\sigma(f)(wN,z) = f\big(\big(z^{-1}N, e, w^{-1}N, wN, z\big)L\big); \tag{5.3}$$

Φ_τ is induced by the K-equivariant isomorphism $\varphi_\tau:\mathrm{Ind}_G^{P\#Q}\tau \to \mathrm{Ind}_L^P\eta$ given by

$$\varphi_\tau(f)(rN,u,vN) = f\big(\big(rN,u,vN,v^{-1}rN,e\big)L\big). \tag{5.4}$$

5.2. $P \# Q$ and R are isomorphic

The map $\psi: P \# Q \to R$ given by

$$\psi\big((rN,u,vN,wN,z)L\big) = (rzN, uz, vH)$$

is a (well-defined) homeomorphism with inverse given by

$$\psi^{-1}(rN,u,vH) = \big(rN,u,vN,v^{-1}rN,e\big)L.$$

Since ψ is equivariant for the left action of K and the right action of G, ψ induces a K-equivariant isomorphism $\psi_\tau:\mathrm{Ind}_G^R\tau \to \mathrm{Ind}_G^{P\#Q}\tau$ such that

$$\psi_\tau(f)\big((rN,u,vN,wN,z)L\big) = f(rzN,uz,vH) \tag{5.5}$$

and a G-equivariant isomorphism $\psi_\sigma:\mathrm{Ind}_K^R\sigma \to \mathrm{Ind}_K^{P\#Q}\sigma$ with the same rule:

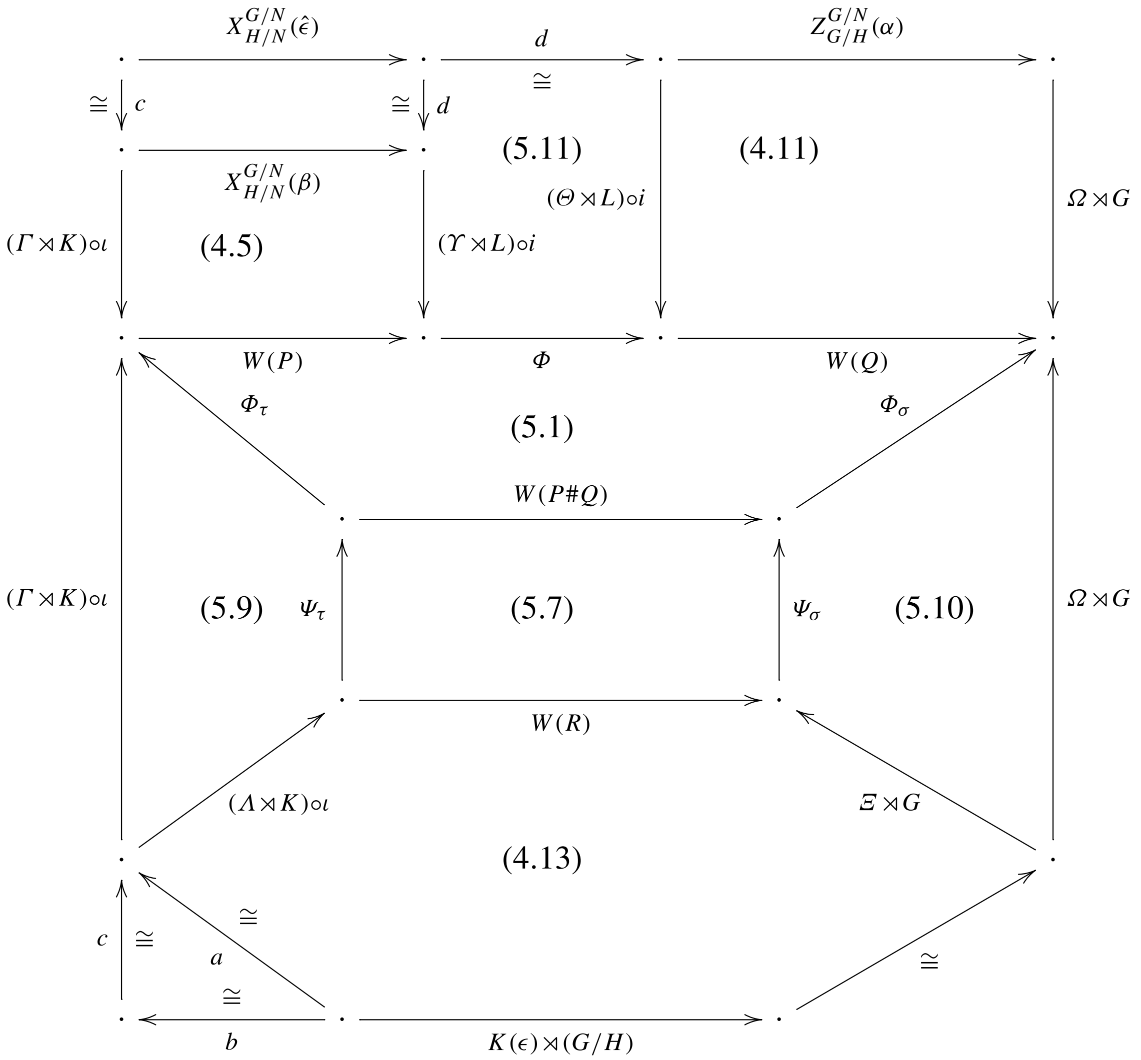
$$\psi_\sigma(f)\big((rN,u,vN,wN,z)L\big) = f(rzN,uz,vH). \tag{5.6}$$

The map of $C_c(R, A)$ into $C_c(P\#Q, A)$ induced by ψ extends to an imprimitivity bimodule isomorphism $\Psi : W(R) \to W(P\#Q)$ whose coefficient maps are $\Psi_\tau := \psi_\tau \rtimes K$ and $\Psi_\sigma := \psi_\sigma \rtimes G$. In other words, the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Ind}_G^R \tau \rtimes_{\sigma\otimes\mathrm{lt}} K & \xrightarrow{\quad W(R) \quad} & \operatorname{Ind}_K^R \sigma \rtimes_{\tau\otimes\mathrm{rt}} G \\ {\scriptstyle \Psi_\tau = \psi_\tau \rtimes K}\ \Big\downarrow \cong & & \cong \Big\downarrow\ {\scriptstyle \Psi_\sigma = \psi_\sigma \rtimes G} \\ \operatorname{Ind}_G^{P\#Q} \tau \rtimes_{\sigma\otimes\mathrm{lt}} K & \xrightarrow{\quad W(P\#Q) \quad} & \operatorname{Ind}_K^{P\#Q} \sigma \rtimes_{\tau\otimes\mathrm{rt}} G. \end{array} \tag{5.7}$$

5.3. *Assembly*

Now we assemble the commuting diagrams involving the three bimodules from Theorem 4.2 into diagram (5.8) below. (For simplicity we only indicate the bimodules and isomorphisms, and the respective diagram numbers.) Note that every arrow is invertible, and the outer rectangle (whose vertical sides collapse) is precisely diagram (4.3). Thus, to complete the proof of Theorem 4.2, it remains to show that the squares labelled (5.9), (5.10), and (5.11) commute, as well as the upper and lower left-hand corners.



(5.8)

5.4. Noncanonical isomorphisms in (5.8)

Using Eqs. (5.4), (5.5), (4.14), and (4.7), for any $f \in A \otimes C_0(G/N) \otimes C_0(G/H)$ and any $(rN, u, vN) \in R$ we have

$$\begin{aligned}
\varphi_\tau\big(\psi_\tau\big(\Lambda(f)\big)\big)(rN, u, vN) &= \psi_\tau\big(\Lambda(f)\big)\big(\big(rN, u, vN, v^{-1}rN, e\big)L\big) \\
&= \Lambda(f)(rN, u, vH) \\
&= f\big(ru^{-1}N, vH\big) \\
&= \Gamma(f)(rN, u, vN).
\end{aligned}$$

So $\varphi_\tau \circ \psi_\tau \circ \Lambda = \Gamma : A \otimes C_0(G/N) \otimes C_0(G/H) \to \operatorname{Ind}_L^R \eta$. Since all four maps are K-equivariant, it follows that the following diagram of isomorphisms commutes:

$$\begin{array}{ccc}
((A \otimes C_0(G/N) \rtimes_{\alpha\otimes\mathrm{lt}} G) \otimes C_0(G/H)) \rtimes_{\beta\otimes\mathrm{lt}} (G/N) & \xrightarrow{(\Lambda\rtimes K)\circ\iota} & \operatorname{Ind}_G^R \tau \rtimes_{\sigma\otimes\mathrm{lt}} K \\
\Big\downarrow{\scriptstyle (\Gamma\rtimes K)\circ\iota} & & \Big\downarrow{\scriptstyle \Psi_\tau=\psi_\tau\rtimes K} \\
\operatorname{Ind}_L^P \eta \rtimes_{\sigma\otimes\mathrm{lt}} K & \xleftarrow{\Phi_\tau=\varphi_\tau\rtimes K} & \operatorname{Ind}_G^{P\#Q} \tau \rtimes_{\sigma\otimes\mathrm{lt}} K.
\end{array} \tag{5.9}$$

Using Eqs. (5.3), (5.6), (4.15), and (4.9), for any $f \in A \otimes C_0(G/H)$ and any $(wN, z) \in Q$ we have

$$\begin{aligned}
\varphi_\sigma\big(\psi_\sigma\big(\Xi(f)\big)\big)(wN, z) &= \psi_\sigma\big(\Xi(f)\big)\big(\big(N, z, w^{-1}N, wN, e\big)L\big) \\
&= \Xi(f)\big(N, z, w^{-1}H\big) \\
&= \alpha_z\big(f\big(w^{-1}H\big)\big) \\
&= \Omega(f)(wN, z).
\end{aligned}$$

Thus $\varphi_\sigma \circ \psi_\sigma \circ \Xi = \Omega : A \otimes C_0(G/H) \to \operatorname{Ind}_L^Q \zeta$. All four maps are G-equivariant, so the following diagram commutes:

$$\begin{array}{ccc}
\operatorname{Ind}_K^R \sigma \rtimes_{\tau\otimes\mathrm{rt}} G & \xleftarrow{\Xi\rtimes G} & (A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\mathrm{lt}} G \\
\Big\downarrow{\scriptstyle \Psi_\sigma=\psi_\sigma\rtimes G} & & \Big\downarrow{\scriptstyle \Omega\rtimes G} \\
\operatorname{Ind}_K^{P\#Q} \sigma \rtimes_{\tau\otimes\mathrm{rt}} G & \xrightarrow{\Phi_\sigma=\varphi_\sigma\rtimes G} & \operatorname{Ind}_L^Q \zeta \rtimes_{\tau\otimes\mathrm{rt}} G.
\end{array} \tag{5.10}$$

For $f \in A \otimes C_0(G/N)$ and any $(rN, s) \in G/N \times G$, using Eqs. (5.2), (4.6) and (4.10), we have

$$T\big(\Upsilon(f)\big)(rN, s) = \Upsilon(f)\big(s^{-1}N, e, r^{-1}N\big) = f\big(sr^{-1}N\big) = \Theta(f)(rN, s),$$

so $T \circ \Upsilon = \Theta$. All three maps are L-equivariant, so the following diagram commutes:

$$
\begin{array}{ccc}
\operatorname{Ind}_K^P \sigma \rtimes_{\eta\otimes\mathrm{rt}} L & \xrightarrow{\Phi = T \rtimes L} & \operatorname{Ind}_G^Q \tau \rtimes_{\zeta\otimes\mathrm{lt}} L \\
\nwarrow{\scriptstyle (\Upsilon\rtimes L)\circ i} & & \nearrow{\scriptstyle (\Theta\rtimes L)\circ i} \\
& ((A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G) \rtimes_{\beta|} (H/N). &
\end{array}
\tag{5.11}
$$

5.5. *Canonical isomorphisms in (5.8)*

For the upper left-hand square of diagram (5.8), temporarily set $C = A \rtimes_\alpha G \rtimes_\epsilon (G/N)$ and $D = (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G$. Then it is straightforward to verify that the $\hat{\epsilon}$–β equivariant canonical map of C onto D induces an imprimitivity bimodule isomorphism of $X_{H/N}^{G/N}(\hat{\epsilon})$ onto $X_{H/N}^{G/N}(\beta)$ such that the diagram

$$
\begin{array}{ccc}
(C \otimes C_0(G/H)) \rtimes_{\hat{\epsilon}\otimes\mathrm{lt}} (G/N) & \xrightarrow{X_{H/N}^{G/N}(\hat{\epsilon})} & C \rtimes_{\hat{\epsilon}|} (H/N) \\
\cong \big\downarrow c & & \cong \big\downarrow d \\
(D \otimes C_0(G/H)) \rtimes_{\beta\otimes\mathrm{lt}} (G/N) & \xrightarrow{X_{H/N}^{G/N}(\beta)} & D \rtimes_{\beta|} (H/N)
\end{array}
\tag{5.12}
$$

commutes. The lower left-hand triangle of (5.8), which is enlarged below, commutes because all the isomorphisms are canonical.

$$
\begin{array}{ccc}
(D \otimes C_0(G/H)) \rtimes_{\beta\otimes\mathrm{lt}} (G/N) & & \\
c \big\uparrow \cong & \nwarrow{\scriptstyle a}\ \cong & \\
(C \otimes C_0(G/H)) \rtimes_{\hat{\epsilon}\otimes\mathrm{lt}} (G/N) & \xleftarrow[b]{\cong} & C \rtimes_{\hat{\epsilon}} (G/N) \rtimes_{\hat{\hat{\epsilon}}|} (G/H).
\end{array}
$$

This completes the proof of Theorem 4.2.

6. Induction in stages

We can deduce induction-in-stages for the Z's from results already in the literature.

Proposition 6.1. *Let $\alpha : G \to \operatorname{Aut} A$ be a continuous action of a locally compact group G by automorphisms of a C^*-algebra A. Also let H and N be closed subgroups of G with N normal in G and $N \subseteq H$. Then the following diagram of right-Hilbert bimodules commutes*:

$$\begin{array}{ccc}
(A \otimes C_0(G)) \rtimes_{\alpha\otimes\mathrm{lt}} G & \xrightarrow{\quad Z^G_{G/H}(\alpha) \quad} & (A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\mathrm{lt}} G \\
{\scriptstyle Z^G_{G/N}(\alpha)}\searrow & & \nearrow{\scriptstyle Z^{G/N}_{G/H}(\alpha)} \\
 & (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G. &
\end{array}$$

Proof. Consider the following diagram, where, as usual, the X's denote Green imprimitivity bimodules:

$$\begin{array}{ccccc}
 & & A \rtimes_{\alpha|} N & & \\
 & \nearrow{\scriptstyle X^G_e(\alpha)\rtimes N} & & \nwarrow{\scriptstyle X^G_N(\alpha)} & \\
(A \otimes C_0(G)) \rtimes_{\alpha\otimes\mathrm{lt}} G \rtimes_{\beta|} N & & \xrightarrow{\quad Z^G_{G/N}(\alpha) \quad} & & (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G \\
\Big\downarrow{\scriptstyle \mathrm{Res}} & & \Big\downarrow{\scriptstyle \mathrm{Res}} & & \Big\downarrow{\scriptstyle Z^{G/N}_{G/H}(\alpha)} \\
 & & A \rtimes_{\alpha|} H & & \\
 & \nearrow{\scriptstyle X^G_e(\alpha)\rtimes H} & & \nwarrow{\scriptstyle X^G_H(\alpha)} & \\
(A \otimes C_0(G)) \rtimes_{\alpha\otimes\mathrm{lt}} G \rtimes_{\beta|} H & & \xrightarrow{\quad Z^G_{G/H}(\alpha) \quad} & & (A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\mathrm{lt}} G.
\end{array} \tag{6.1}$$

Commutativity of the right rear face is exactly [11, Proposition 3.5]. The upper and lower (triangular) faces commute by [4, Theorem 3.1].[2] The left rear face commutes by naturality of restriction [14, Lemma 5.7]. Since all except the vertical arrows are imprimitivity bimodules, it follows that the front face commutes.

The commutative front face of diagram (6.1) should be viewed as a strong version of induction in stages; the proposition follows from this because

$$\mathrm{Res} \otimes_{((A\otimes C_0(G))\rtimes_{\alpha\otimes\mathrm{lt}} G)\rtimes_{\beta|} H} Z^G_{G/H}(\alpha) \cong Z^G_{G/H}(\alpha)$$

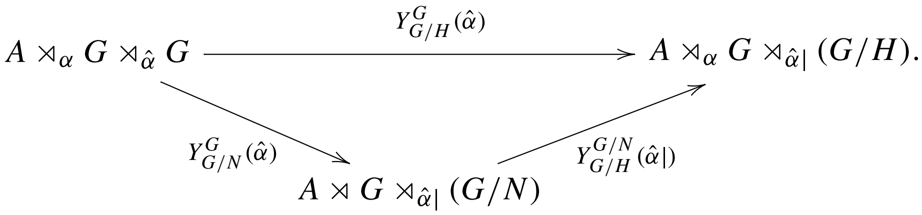
as a right-Hilbert $(A \otimes C_0(G)) \rtimes_{\alpha\otimes\mathrm{lt}} G - (A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\mathrm{lt}} G$ bimodule. □

We next deduce induction-in-stages for the Mansfield bimodule in the case of a dual coaction. The hypotheses are the same as in Proposition 6.1.

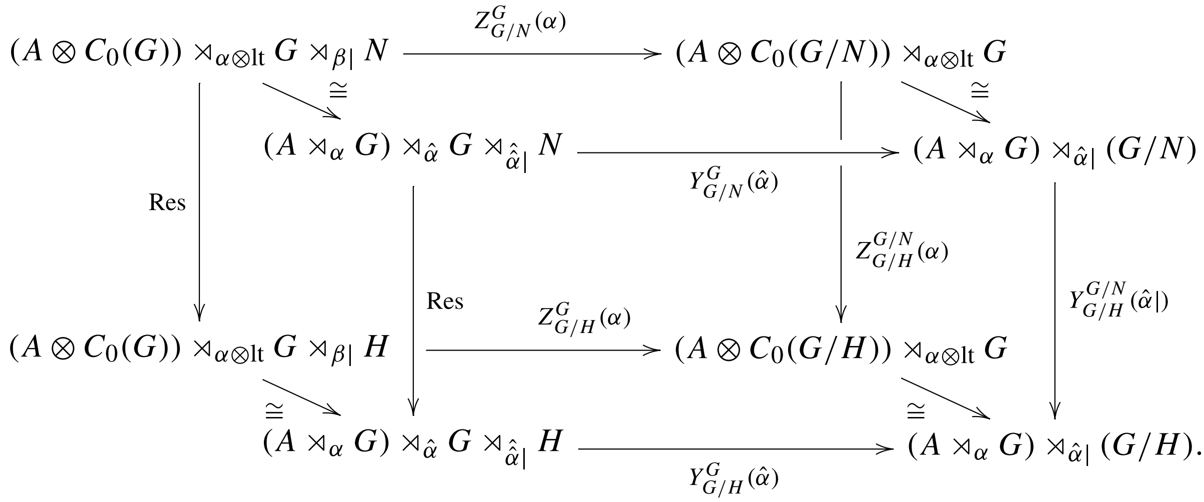
---

[2] The statement of [4, Theorem 3.1] should end with "$- A \rtimes_{\alpha|} H$ bimodules."

Proposition 6.2. *Let $\alpha : G \to \operatorname{Aut} A$ be a continuous action of a locally compact group G by automorphisms of a C^*-algebra A. Also let H and N be closed subgroups of G with N normal in G and $N \subseteq H$. Then the following diagram of right-Hilbert bimodules commutes*:



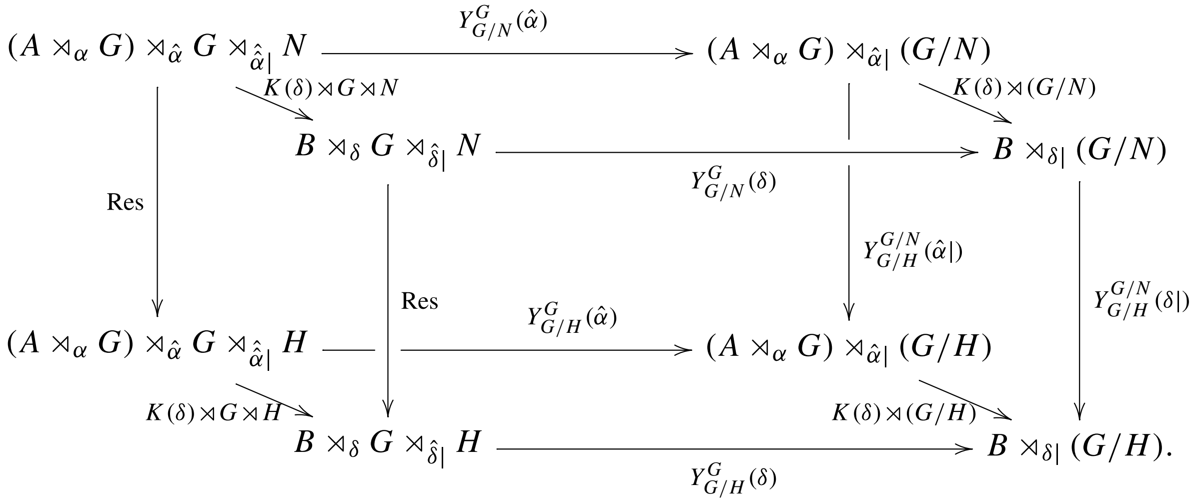
Proof. Consider the following diagram, where the isomorphisms are the canonical ones:



(6.2)

The rear face is the commutative front face of diagram (6.1); the upper and lower faces commute by [4, Proposition 1.1]; the right-hand face is seen to commute by ignoring the left H/N-actions in Theorem 4.1; and it is straightforward to verify directly that the left-hand face commutes (or one can use naturality of restriction [14, Lemma 5.7]). It follows that the front face commutes, and the proposition follows from this as in the proof of Proposition 6.1. □

Proof of Theorem 1.1. Recall that we assume δ is a maximal coaction of a locally compact group G on a C^*-algebra B, and that N and H are closed normal subgroups of G with $N \subseteq H$. Now let $(A, \alpha) = (B \rtimes_\delta G, \hat\delta)$, and consider the following diagram:



(6.3)

The rear face is the commutative front face of diagram (6.2); the upper, lower, and right-hand faces all commute by naturality of the Mansfield bimodule [13, Theorem 6.6]; the left-hand face commutes by naturality of restriction [14, Lemma 5.7]. The arrows connecting the rear face to the front face are all imprimitivity bimodules, hence invertible; it follows that the front face commutes, and the theorem follows from this as in the proof of Proposition 6.1. □

Remark 6.3. The overall structure of our proof of Theorem 1.1 has been: using naturality to pass to dual coactions (diagram (6.3)); in the dual case replacing Mansfield bimodules by symmetric-imprimitivity bimodules (diagram (6.2)); and proving induction-in-stages for symmetric-imprimitivity bimodules directly (diagram (6.1)).

This amounts to gluing these three diagrams together along their common faces, and in fact we might have saved some work by addressing the glued-together diagram directly rather than the three separate pieces. For example, part of the top face of the glued-together diagram would be

$$\begin{array}{ccc}
(A \otimes C_0(G)) \rtimes_{\alpha\otimes \mathrm{lt}} G \rtimes_{\beta|} N & \xrightarrow{Z^G_{G/N}(\alpha)} & (A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\mathrm{lt}} G \\
\cong \downarrow & & \downarrow \cong \\
(A \rtimes_\alpha G) \rtimes_{\hat\alpha} G \rtimes_{\hat\alpha|} N & \xrightarrow{Y^G_{G/N}(\hat\alpha)} & A \rtimes_\alpha G \rtimes_{\hat\alpha|} (G/N) \\
K(\delta)\rtimes G \rtimes N \downarrow & & \downarrow K(\delta) \rtimes (G/N) \\
B \rtimes_\delta G \rtimes_{\hat\delta|} N & \xrightarrow{Y^G_{G/N}(\delta)} & B \rtimes_{\delta|} (G/N),
\end{array} \tag{6.4}$$

and the outer square of (6.4) is already known to commute: it is precisely the definition of the Mansfield bimodule $Y^G_{G/N}(\delta)$ [13, Theorem 5.3]. While the argument may have been have made shorter in this way, we feel that it is much better understood in terms of the three separate pieces.

For future reference, we state as a corollary of the proof of Theorem 1.1 the strong version of induction in stages which appears in diagram (6.3). This is the analogue for maximal coactions of Theorem 4.1 of [14], which was proved for a (not-necessarily-maximal) coaction δ of G on B and normal subgroups $N \subseteq H$ of G such that "Mansfield imprimitivity works for H."

Corollary 6.4. *Let $\delta : B \to M(B \otimes C^*(G))$ be a maximal coaction of a locally compact group G on a C^*-algebra B. Also let N and H be closed normal subgroups of G such that $N \subseteq H$. Then the diagram*

$$\begin{array}{ccc} B \rtimes_\delta G \rtimes_{\hat{\delta}|} N & \xrightarrow{Y^G_{G/N}(\delta)} & B \rtimes_{\delta|} (G/N) \\ {\scriptstyle \mathrm{Res}}\big\downarrow & & \big\downarrow {\scriptstyle Y^{G/N}_{G/H}(\delta|)} \\ B \rtimes_\delta G \rtimes_{\hat{\delta}|} H & \xrightarrow[Y^G_{G/H}(\delta)]{} & B \rtimes_{\delta|} (G/H) \end{array} \tag{6.5}$$

of right-Hilbert bimodules commutes.

7. Another application of Theorem 3.1

Consider symmetric imprimitivity data $({}_K X_G, A, \sigma, \tau)$. Then $({}_{\{e\}} X_G, A, \tau)$ is valid data as well, and $W({}_{\{e\}} X_G)$ is an $\mathrm{Ind}_G^X \tau$–$(C_0(X, A) \rtimes_{\tau\otimes\mathrm{rt}} G)$-imprimitivity bimodule which carries an action $(\sigma \otimes \mathrm{lt}, \sigma \otimes \mathrm{lt}, (\sigma \otimes \mathrm{lt}) \rtimes \mathrm{id})$ of K. Taking the crossed product of $W({}_{\{e\}} X_G)$ by the action of K (see [1,2]) we get an

$$\big(\mathrm{Ind}_G^X \tau \rtimes_{\sigma\otimes\mathrm{lt}} K\big)\text{–}\big(\big(C_0(X, A) \rtimes_{\tau\otimes\mathrm{rt}} G\big) \rtimes_{\sigma\otimes\mathrm{lt}\rtimes\mathrm{id}} K\big)$$

imprimitivity bimodule $W({}_{\{e\}} X_G) \rtimes_{\sigma\otimes\mathrm{lt}} K$ which is a completion of $C_c(K, C_c(X, A))$. Similarly, $W({}_K X_{\{e\}})$ carries an action $((\tau \otimes \mathrm{rt}) \rtimes \mathrm{id}, \tau \otimes \mathrm{rt}, \tau \otimes \mathrm{rt})$ of G, and taking crossed products by G gives an

$$\big(\big(C_0(X, A) \times_{\sigma\otimes\mathrm{lt}} K\big) \rtimes_{\tau\otimes\mathrm{rt}\rtimes\mathrm{id}} G\big)\text{–}\big(\mathrm{Ind}_K^X \sigma \rtimes_{\tau\otimes\mathrm{rt}} G\big)$$

imprimitivity bimodule $W({}_K X_{\{e\}}) \rtimes_{\tau\otimes\mathrm{rt}} G$. Let

$$\Psi : \big(C_0(X, A) \rtimes_{\sigma\otimes\mathrm{lt}} K\big) \rtimes_{\tau\otimes\mathrm{rt}\rtimes\mathrm{id}} G \to \big(C_0(X, A) \rtimes_{\tau\otimes\mathrm{rt}} G\big) \rtimes_{\sigma\otimes\mathrm{lt}\rtimes\mathrm{id}} K$$

be the natural isomorphism. It was proved in [9, Lemma 4.8] that there is an imprimitivity bimodule isomorphism

$$\big(W({}_{\{e\}} X_G) \rtimes K\big) \otimes_\Psi \big(W({}_K X_{\{e\}}) \rtimes G\big) \cong W({}_K X_G), \tag{7.1}$$

and it is an obvious test question for Theorem 3.1 whether it can recover this isomorphism on the level of spaces.

The first step is to note that $W({}_{\{e\}} X_G) \rtimes_{\sigma\otimes\mathrm{lt}} K$ is isomorphic to the imprimitivity bimodule $W({}_K P_{K\times G}, A, \mathrm{id}, \sigma \times \tau)$ where $P := K \times X$ and

$$k \cdot (t, x) = (kt, x) \quad \text{and} \quad (t, x) \cdot (k, m) = \big(tk, k^{-1} \cdot x \cdot m\big).$$

To see this, note that the map $K(t,x)\mapsto x$ is a homeomorphism of K/P onto X and $(t,x)(K\times G)\mapsto t\cdot xG$ is a homeomorphism of $P/(G\times K)$ onto X/G, and define

$$\Lambda : \operatorname{Ind}_G^X \tau \to \operatorname{Ind}_{K\times G}^P(\sigma\times\tau) \quad \text{by } \Lambda(f)(t,x)=\sigma_t^{-1}\big(f(t\cdot x)\big);$$
$$\Theta : C_0(X,A)\to \operatorname{Ind}_K^P \mathrm{id} \quad \text{by } \Theta(h)(t,x)=h(x).$$

It is easy to check that Λ and Θ are well defined and invertible, with inverses given by

$$\Lambda^{-1}(g)(x)=g(e,x) \quad\text{and}\quad \Theta^{-1}(l)(e,x)=l(e,x)$$

for $g\in \operatorname{Ind}_{K\times G}^P(\sigma\times\tau)$ and $l\in\operatorname{Ind}_K^P \mathrm{id}$. To check that Λ is equivariant for the actions of K, it helps to write lt^X and lt^P to distinguish between actions induced from left actions on different spaces. Then,

$$\begin{aligned}
\Lambda\big(\big(\sigma\otimes\mathrm{lt}^X\big)_k(f)\big)(t,x) &= \sigma_t^{-1}\big(\big(\sigma\otimes\mathrm{lt}^X\big)_k(f)(t\cdot x)\big)\\
&= \sigma_t^{-1}\sigma_k\big(f\big(k^{-1}t\cdot x\big)\big)\\
&= \Lambda(f)\big(k^{-1}t,x\big)\\
&= \Lambda(f)\big(k^{-1}\cdot(t,x)\big)\\
&= \big(\mathrm{id}\otimes\mathrm{lt}^P\big)_k\big(\Lambda(f)\big)(t,x).
\end{aligned}$$

Similarly, Θ is $((\tau\times\sigma)\otimes(\mathrm{rt}^X\times\mathrm{lt}^X))$–$((\sigma\times\tau)\otimes\mathrm{rt}^P)$ equivariant. Thus Λ and Θ induce isomorphisms

$$\Lambda\rtimes K : \operatorname{Ind}_G^X\tau\rtimes_{\sigma\otimes\mathrm{lt}^X}K\to\operatorname{Ind}_{K\times G}^P(\sigma\times\tau)\rtimes_{\mathrm{id}\otimes\mathrm{lt}^P}K,$$
$$\Theta\rtimes(G\times K) : C_0(X,A)\rtimes_{(\tau\times\sigma)\otimes(\mathrm{rt}^X\times\mathrm{lt}^X)}(G\times K)\to\operatorname{Ind}_K^P\mathrm{id}\rtimes_{(\sigma\times\tau)\otimes\mathrm{rt}^P}(K\times G).$$

For $z\in C_c(K,C_c(X,A))$ define

$$\Upsilon(z)(t,x)=\sigma_t^{-1}\big(z(t)(t\cdot x)\big)\Delta_K(t)^{1/2}.$$

It is not hard to check, using the formulas given at [6, Eqs. (B.2)] for the symmetric imprimitivity theorem bimodules and at [10, Eqs. 3.5–3.8] for the Combes crossed product, that $(\Lambda\rtimes K,\Upsilon,\Theta\rtimes(G\times K))$ extends to an imprimitivity bimodule isomorphism of $W({}_{\{e\}}X_G)\times_{\sigma\otimes\mathrm{lt}}K$ onto $W({}_K P_{K\times G})$.

Similarly, $W({}_K X_{\{e\}})\rtimes_{\tau\otimes\mathrm{rt}}G$ is isomorphic to the imprimitivity bimodule associated to the data $({}_{K\times G}Q_G, A, \sigma\times\tau, \mathrm{id})$ where $Q:=G\times X$ and

$$(s,x)\cdot m=\big(m^{-1}s,x\big) \quad\text{and}\quad (k,m)\cdot(s,x)=\big(sm^{-1},k\cdot x\cdot m^{-1}\big).$$

(In place of $(\Lambda,\Upsilon,\Theta)$ use (Γ,Ω,Ξ) where, for $s\in G$,

$$\Gamma : \operatorname{Ind}_K^X \sigma \to \operatorname{Ind}_{K\times G}^Q(\sigma\times\tau) \text{ is } \Gamma(f)(s,x) = \tau_s^{-1}\big(f\big(x\cdot s^{-1}\big)\big);$$
$$\Xi : C_0(X,A) \to \operatorname{Ind}_G^Q \mathrm{id} \text{ is } \Xi(h)(s,x) = h(x); \quad \text{and}$$
$$\Omega : C_c\big(G, C_c(X,A)\big) \to W(Q) \text{ is } \Omega(z)(s,x) = z\big(s^{-1},x\big)\Delta_G(s)^{-1/2}.)$$

The hypotheses of Theorem 3.1 are satisfied with $\varphi : K \setminus P \to Q/G$ given by $\varphi(K(t,x)) = (e,x)G$ and $\tilde\sigma_{(t,x)} = \sigma_t$ and $\tilde\tau_{(s,x)} = \tau_s$. Thus

$$\begin{aligned} P \times_\varphi Q &= \big\{(t,x,s,y)\colon\ t\in K,\ s\in G,\ x,y\in X \text{ and } \varphi\big(K(t,x)\big) = (s,y)G\big\} \\ &= \big\{(t,x,s,x)\colon\ t\in K,\ s\in G,\ x\in X\big\} \end{aligned}$$

and $K\times G$ acts on $P\times_\varphi Q$ by the diagonal action

$$(t,x,s,x)\cdot(k,m) = \big(tk, k^{-1}\cdot x\cdot m, sm, k^{-1}\cdot x\cdot m\big).$$

The map $\psi : P\times_\varphi Q \to X$ given by $(t,x,s,x)\mapsto t\cdot x\cdot s^{-1}$ induces a homeomorphism $\bar\psi$ of $P\#Q = (P\times_\varphi Q)/(K\times G)$ onto X. Then $\bar\psi$ is equivariant for the actions of K and G because ψ is: for $k\in K$ and $m\in G$ we have

$$\begin{aligned} k\cdot\psi(t,x,s,x) &= k\cdot\big(t\cdot x\cdot s^{-1}\big) = kt\cdot x\cdot s^{-1} = \psi(kt,x,s,x) = \psi\big(k\cdot(t,x,s,x)\big) \\ \psi(t,x,s,x)\cdot m &= \big(t\cdot x\cdot s^{-1}\big)\cdot m = t\cdot x\cdot s^{-1}m = \psi\big(t,x,m^{-1}s\big) = \psi\big((t,x,s)\cdot m\big). \end{aligned}$$

Thus $W({}_K(P\#Q)_G)$ and $W({}_K X_G)$ are isomorphic. The isomorphism (7.1) now follows from Theorem 3.1.

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