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Multiply Perfect Numbers, Mersenne Primes, and Effective Computability*

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§ 1. Introduction

In this paper we shall consider the equation

$$\sigma(n)/n = \alpha \tag{1.1}$$

where α is an arbitrary fixed rational and σ is the sum of the divisors function. Special cases of (1.1) have been extensively studied since antiquity. In particular, the case $\alpha = 2$ corresponds to perfect numbers, and the case when α is an integer corresponds to multiply perfect numbers. It is not known if for any α whether (1.1) has infinitely many solutions. However Hornfeck and Wirsing [11] have shown that the number of solutions $n \leqq x$ of (1.1) is $o(x^\varepsilon)$ for any $\varepsilon > 0$ uniformly for all α.

We recall the Euclid-Euler formula for even perfect numbers: an even integer n is perfect if and only if $n = 2^{k-1}(2^k - 1)$ where $2^k - 1$ is a (Mersenne) prime. Hence the question of the infinitude of even perfect numbers is equivalent to the question of the infinitude of the Mersenne primes (only 24 are known to date).

In this paper we shall call a natural number n *non-primitive* if $n = em$ where e is an even perfect number and $(e, m) = 1$. Otherwise we shall call n *primitive*. If there are infinitely many Mersenne primes and if m is odd and satisfies $\sigma(m)/m = \alpha/2$, then there are infinitely many non-primitive solutions of (1.1) with a fixed number of distinct prime factors (namely, the integers em where e runs over the even perfect numbers relatively prime to m). However primitive solutions of (1.1) provide a different story:

Theorem (Kanold [12]). *For any α there are only finitely many primitive solutions of (1.1) with a fixed number of distinct prime factors.*

Kanold's theorem has an interesting corollary, namely that if for some arbitrary choice for α, K there are infinitely many solutions of (1.1) with exactly K distinct prime factors, then there also must be infinitely many Mersenne primes. Indeed, the various non-primitive solutions of (1.1) can involve only finitely many odd (and

* In memory of C. W. Anderson (1945—1974)

hence primitive) integers m with exactly $K-2$ distinct prime factors and satisfying $\sigma(m)/m=\alpha/2$.

Kanold's proof uses the fact that $ax^3-by^3=c$ has only finitely many integer solutions x, y for given positive integers a, b, c. Considering the recent work of Baker and others on effective solvability of some Diophantine equations (and in particular, all the equations in the family Kanold considers) one might expect that an effective version of Kanold's theorem can be proved. This is in fact the case.

In this paper we shall provide a constructive proof of Kanold's theorem. We shall do this by giving an algorithm by which a number $N(\alpha, K)$ may be found such that all primitive solutions of (1.1) with precisely K distinct prime factors are bounded above by $N(\alpha, K)$.

The main tool for a crucial step in our proof is a deep theorem of Baker [5] (see § 2). While effective, the bound in Baker's result is very large. Moreover the algorithm we provide for the construction of $N(\alpha, K)$ could conceivably pass through the step requiring Baker's theorem many times. Consequently, in our opinion, any explicit formula for $N(\alpha, K)$ achieved by the methods in this paper would be too large to be of any interest. (It would be larger than $\exp_K(K)$ where $\exp_K$ denotes K iterations of exp.) However for odd solutions of (1.1) we do not need Baker's theorem, and thus we can establish a somewhat lower bound. In fact an upper bound for the odd solutions of (1.1) which have exactly K distinct prime factors is

$$(2cK)^{(2cK)^{2^{K^2}}}, \tag{1.2}$$

where c is the numerator of α in reduced form (see § 5).

We remark that Kanold's theorem for the case $\alpha=2$ was previously proved by Dickson [9] and Gradstein [10].

Our algorithm can circumvent Baker's theorem in the special case $\alpha=3$ by using a result of Steuerwald [16]: if $\sigma(n)/n=3$, then $36 \nmid n$. However Steuerwald's method does not appear to generalize to other cases.

Recently Artuhov [4] and Borho [7] have obtained results for amicable numbers similar to Kanold's theorem. In addition Borho [8] has obtained an explicit bound for all amicable number pairs u, v for which uv has a fixed number of prime factors (not necessarily distinct).

C. W. Anderson inspired much of the work for this paper. In particular, his problem [1] introduced me to the question of saying something about the set of rationals of the form $\sigma(n)/n$. In [2] Anderson conjectured that the set of rationals of the form $\sigma(n)/n$ is recursive. He showed how this conjecture is related to the odd perfect number problem; for example, if $5/3$ can be written in the form $\sigma(n)/n$, then $5n$ must be an odd perfect number. In § 6 we shall give a sufficient (and perhaps necessary) condition for a weaker form of Anderson's conjecture to hold.

§ 2. Baker's Theorem

The theorem to which we refer deals with integral linear combinations of logarithms of $n \geqq 2$ algebraic numbers. We will use this result for the special case $n=3$ and the algebraic numbers positive rationals:

Theorem (Baker [5]). *Suppose* $0<\delta\leqq 1$. *If* $\alpha_1, \alpha_2, \alpha_3$ *are positive rational numbers with their numerators and denominators at most* $A\geqq 4$, *and if* b_1, b_2, b_3 *are integers with absolute values at most* H, *then*

$$0<|b_1\log\alpha_1+b_2\log\alpha_2+b_3\log\alpha_3|<e^{-\delta H},$$

implies

$$H<2^{1470}\delta^{-49}(\log A)^{49}.$$

The bound in this theorem is not as sharp as possible; indeed it is known that $H=0((\log A)^{3+\varepsilon})$ for every $\varepsilon>0$. However the effectively computable bounds we obtain in this paper are impractically large even for odd solutions of (1.1) [see (1.2)], a case where we do not need Baker's theorem. Since our results are impractical for actually computing solutions of (1.1) in the easy (i.e., odd) case, we will not split hairs in the hard case.

§ 3. Preliminaries

If n is a rational number, we let $h(n)=\sigma(n)/n$. Then h is multiplicative, and if $m|n$, $m\neq n$, we have $h(m)<h(n)$. If p is a prime, then $h(p^a)$ increases with a to the limiting value $p/(p-1)$. Hence we define $h(p^\infty)=p/(p-1)$. If $p_1, \ldots, p_k$ are distinct primes, then by $h(\Pi p_i^\infty)$ we mean $\Pi h(p_i^\infty)=\Pi p_i/(p_i-1)$.

If p is a prime, denote by $v_p(n)$ the exponent (possibly 0) appearing on p in the prime factorization of n. We define $\omega(n)=\Sigma_{p|n}1$, $\Omega(n)=\Sigma_{p|n}v_p(n)$. Let $\tau(n)$ denote the divisor function, so $\tau(n)=\Pi_{p|n}(1+v_p(n))$.

Using the equation $\sigma(q^a)=(q^{a+1}-1)/(q-1)$ for primes q, and Theorems 94 and 95 in Nagell [13], we have

Lemma 3.1. *If* $p\neq q$ *are primes, then*

$$v_p(\sigma(q^a))\leqq\begin{cases}v_p(\sigma(q^{p-2}))+v_p(a+1), & \text{if} \quad p>2\\ v_2(q+1)+v_2(a+1)-1, & \text{if} \quad p=2.\end{cases}$$

From Bang [6] and many others we may obtain

Lemma 3.2. *If* q *is a prime, then* $\omega(\sigma(q^a))\geqq\tau(a+1)-2$.

§ 4. The Main Result

In this section we shall prove

Theorem 4.1. *For every rational* $\alpha\geqq 1$ *and every non-negative integer* K, *there is an effectively computable number* $N(\alpha, K)$ *such that if* $\omega(n)=K$ *and* n *is a primitive solution of* (1.1), *then* $n\leqq N(\alpha, K)$.

We introduce some notation. If m, n are positive integers, we shall write $m|^*n$ if $m|n$ and every prime factor of m is less than every prime factor of n/m. For example, $1|^*12$, $4|^*12$, $12|^*12$, but $2\nmid^*12$, $3\nmid^*12$, $6\nmid^*12$.

Suppose

c, d are positive integers with $(c,d)=1$;

$\alpha=c/d$;

K, k are integers with $K \geqq k \geqq 0$;

$p_1, \ldots, p_k$ are primes with $p_1 < \ldots < p_k$.

Let

$A(\alpha)=\{n : n \text{ is primitive and } \sigma(n)/n=\alpha\}$

$A(\alpha, K)=\{n \in A(\alpha) : \omega(n)=K\}$

$A(\alpha, K, k)=\{m : \omega(m)=k \text{ and } m|^*n \text{ for some } n \in A(\alpha, K)\}$

$P(\alpha, K, k)=\sup\{p \text{ prime} : p|m \text{ for some } m \in A(\alpha, K, k)\}$

$R(\alpha, K, k)=\inf\{\alpha-h(m) : m \in A(\alpha, K, k)\}$

$B(\alpha, K; p_1, \ldots, p_k)=\left\{m \in A(\alpha, K, k): \prod_{i=1}^{k} p_i|m\right\}$

$S(\alpha, K; p_1, \ldots, p_k)=\inf\{\alpha-h(m) : m \in B(\alpha, K; p_1, \ldots, p_k)\}$.

Our first task will be to obtain effectively computable upper bounds for the $P(\alpha, K, k)$.

Theorem 4.2. *For every choice of* α, K, k *with* $K > k \geqq 0$, *and primes* $p_1 < \ldots < p_k$, *there are effectively computable positive constants* $p(\alpha, K, k+1), r(\alpha, K, k), s(\alpha, K; p_1, \ldots, p_k)$ *with*

$p(\alpha, K, k+1) \geqq P(\alpha, K, k+1)$

$r(\alpha, K, k) \leqq R(\alpha, K, k)$

$s(\alpha, K; p_1, \ldots, p_k) \leqq S(\alpha, K; p_1, \ldots, p_k)$.

Proof. First we note that if we can effectively compute $p(\alpha, K, k)$ and $s(\alpha, K; p_1, \ldots, p_k)$ for all choices of primes $p_1 < \ldots < p_k$, we may choose $r(\alpha, K, k)$ as any number satisfying the inequality

$$0 < r(\alpha, K, k) \leqq \inf\{s(\alpha, K; p_1, \ldots, p_k): p_k \leqq p(\alpha, K, k)\}, \tag{4.1}$$

so that $r(\alpha, K, k)$ is also effectively computable.

We now show that we may take

$$p(\alpha, K, k+1)=(K-k)\,\alpha/r(\alpha, K, k), \tag{4.2}$$

so that if $r(\alpha, K, k)$ is effectively computable, so is $p(\alpha, K, k+1)$. Indeed, suppose $m \in A(\alpha, K, k+1)$, p is the largest prime factor of m, and $m=m'p^a$ where $m'|^*m$. Then $m' \in A(\alpha, K, k)$. Also $\alpha < h(m')\,h(p^\infty)^{K-k}$, so that

$h(m')/\alpha > (1-1/p)^{K-k} \geqq 1-(K-k)/p$.

Hence

$$p<\frac{(K-k)\alpha}{\alpha-h(m')}\leqq\frac{(K-k)\alpha}{r(\alpha,K,k)},$$

which proves (4.2).

Next we note that we may assume

$$c>\sigma(d). \tag{4.3}$$

Indeed it is easy to show that if $c<\sigma(d)$, then $A(\alpha)=\emptyset$; and if $c=\sigma(d)$, then $A(\alpha)=\{d\}$. Thus if $c\leqq\sigma(d)$, everything is effectively computable, so we may assume (4.3). Thus $\alpha\neq 1$ and $\alpha\neq\sigma(p^a)/p^a$ for any prime p and positive integer a. These facts imply

$$A(\alpha,0)=\emptyset \quad\text{and}\quad A(\alpha,1)=\emptyset,$$

so that Theorem 4.2 is proved for $K=0$ and 1.

Suppose now $K\geqq 2$. It is clear that we may take

$$r(\alpha,K,0)=\alpha-1$$

so that from (4.2) we may take

$$p(\alpha,K,1)=\frac{K\alpha}{\alpha-1}<cK. \tag{4.4}$$

We now compute $s(\alpha,K;p)$ for an arbitrary prime p. If $h(p^\infty)<\alpha$ and if $p^a\in B(\alpha,K;p)$, then

$$\alpha-h(p^a)>\alpha-h(p^\infty)=\frac{c}{d}-\frac{p}{p-1}\geq\frac{1}{d(p-1)}.$$

If $h(p^\infty)>\alpha$ and if $p^a\in B(\alpha,K;p)$, then

$$\begin{aligned}\alpha-h(p^a)&\geqq\frac{1}{dp^a(p-1)}=\frac{1}{d}(h(p^\infty)-h(p^a))\\&>\frac{1}{d}(h(p^\infty)-\alpha)\geqq\frac{1}{d^2(p-1)}.\end{aligned}$$

If $h(p^\infty)=\alpha$, then (4.3) implies $p=\alpha=2$. But $A(2)$ is the set of odd perfect numbers, so $B(2,K;2)=\emptyset$.

Hence for every prime p, we may take

$$s(\alpha,K;p)=\frac{1}{d^2(p-1)}. \tag{4.5}$$

Hence (4.1) implies we may take

$$r(\alpha,K,1)=\frac{1}{d^2(p(\alpha,K,1)-1)}.$$

Hence (4.2) and (4.4) imply

$$p(\alpha,K,2)=(K-1)\alpha d^2(p(\alpha,K,1)-1)<KcdcK<c^3K^2. \tag{4.6}$$

We have now proved the theorem for $k<2$. Hence we assume $K>k\geqq 2$, and make the inductive assumption that $p(\alpha, K, k)$ is effectively computable and $r(\beta, K-1, k-1)$ is effectively computable for any β. We shall show now how to effectively compute $s(\alpha, K; p_1, \ldots, p_k)$ for any primes $p_1<\ldots<p_k$. Hence using (4.1) and (4.2) we shall be able to effectively compute $r(\alpha, K, k)$ and $p(\alpha, K, k+1)$. Our theorem will thus follow by induction.

Let $p_1<\ldots<p_k$ be arbitrary primes. We shall compute $s(\alpha, K; p_1, \ldots, p_k)$ in each of the three cases $h(\Pi p_i^\infty)<\alpha$, $h(\Pi p_i^\infty)>\alpha$, and $h(\Pi p_i^\infty)=\alpha$.

Suppose $h(\Pi p_i^\infty)<\alpha$ and $m=\Pi p_i^{a_i}\in B(\alpha, K; p_1, \ldots, p_k)$. Then

$$\alpha-h(m)>\alpha-h(\Pi p_i^\infty)\geqq 1/d\Pi(p_i-1),$$

so that in this case we may take

$$s(\alpha, K; p_1, \ldots, p_k)=\frac{1}{d\Pi(p_i-1)}. \tag{4.7}$$

Suppose $h(\Pi p_i^\infty)>\alpha$. Let

$$x_j=[\log(2kc\Pi(p_i-1))/\log p_j]$$

for $j=1, \ldots, k$ where [ ] is the greatest integer function. Then $p_j^{x_j+1}>2kc\Pi(p_i-1)$. Let $\varepsilon=1/c\Pi(p_i-1)$. Using the inequality $\Pi(1-t_i)\geqq 1-\Sigma t_i$ for positive quantities $t_1, \ldots, t_k$, we have

$$\begin{aligned} h\left(\Pi p_i^{x_i}\right) &= \Pi\frac{p_i^{x_i+1}-1}{(p_i-1)p_i^{x_i}}=\Pi\frac{p_i}{p_i-1}\Pi\left(1-\frac{1}{p_i^{x_i+1}}\right) \\ &\geqq \left(\Pi\frac{p_i}{p_i-1}\right)\left(1-\Sigma\frac{1}{p_i^{x_i+1}}\right) \\ &> \left(\alpha+\frac{1}{d\Pi(p_i-1)}\right)\left(1-\frac{1}{2c\Pi(p_i-1)}\right)=\alpha(1+\varepsilon)\left(1-\frac{1}{2}\varepsilon\right)>\alpha. \end{aligned}$$

Hence if $m=\Pi p_i^{a_i}\in B(\alpha, K; p_1, \ldots, p_k)$, then at least one $a_i<x_i$. But

$$m/p_i^{a_i}\in B(\alpha/h(p_i^{a_i}), K-1; p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_k).$$

Thus we may take

$$s(\alpha, K; p_1, \ldots, p_k)=\inf\{h(p_i^{a_i})\cdot r(\alpha/h(p_i^{a_i}), K-1, k-1): 1\leqq i\leqq k, 1\leqq a_i<x_i\}. \tag{4.8}$$

Our last case is $h(\Pi p_i^\infty)=\alpha$. In this case we have $c/d=\alpha=\Pi p_i/(p_i-1)$, so that $c|\Pi p_i$. Say $m=\Pi p_i^{a_i}\in B(\alpha, K; p_1, \ldots, p_k)$. For $i=1, \ldots, k$ we write

$$\sigma(p_i^{a_i})=u_i v_i \tag{4.9}$$

where every prime factor of u_i is in $\{p_1, \ldots, p_k\}$ and $(v_i, \Pi p_j)=1$. Say $m|^*n$ where $n\in A(\alpha, K)$. Suppose $2\nmid n$. Then $2<p_1<\ldots<p_k$, so d, the denominator of $\alpha=\Pi p_i/(p_i-1)=\sigma(n)/n$, is even. But $d|n$, contradicting $2\nmid n$. Hence we must have $2|n$, so that

$$p_1=2. \tag{4.10}$$

Now $d\sigma(n)=cn$, $c|\Pi p_i$, and $p_i \nmid \sigma(p_i^{a_i})$, so that for $i=1,\ldots,k$, we have

$$\omega(\sigma(p_i^{a_i})) \leqq \omega(n)-1=K-1.$$

Using Lemma 3.2 we then have

$$\tau(a_i+1)\leqq K+1. \tag{4.11}$$

Noting that

$$v_{p_j}(\sigma(p_i^{p_j-2}))\leqq \log_{p_j}(\sigma(p_i^{p_j-2}))<(p_j-1)\log p_i/\log p_j \tag{4.12}$$

for $j>1$ and that

$$v_2(p_i+1)\leqq \log_2(p_i+1)<1+\log p_i/\log 2, \tag{4.13}$$

we have for $i=1,\ldots,k$ [using Lemma 3.1 and (4.11)]

$$\begin{aligned}\Omega(u_i)&=\sum_{j\neq i} v_{p_j}(\sigma(p_i^{a_i}))<\sum_{j\neq i} v_{p_j}(a_i+1)+\sum_{j\neq i}(p_j-1)\log p_i/\log p_j\\ &\leqq \tau(a_i+1)-1+(k-1)(p_k-1)\log p_i/\log p_k\\ &\leqq K+(k-1)(p_k-1).\end{aligned}$$

Hence

$$u_i\leqq p_k^{K-1+(k-1)(p_k-1)}. \tag{4.14}$$

Let

$$D=2^{1498}([K+(k-1)(p_k-1)]\log p_k)^{49}. \tag{4.15}$$

We shall show in the following that at least one $a_j<D$. It will then follow that

$$\begin{aligned}\alpha-h(\Pi p_i^{a_i})&=\Pi h(p_i^{\infty})-\Pi h(p_i^{a_i})\\ &>\Big(\prod_{i\neq j} h(p_i^{\infty})\Big)(h(p_j^{\infty})-h(p_j^{a_j}))\\ &>h(p_j^{\infty})-h(p_j^{a_j})>p_j^{-(a_j+1)}\geqq p_j^{-D}.\end{aligned}$$

Thus we will be able to take

$$s(\alpha,K;p_1,\ldots,p_k)=p_k^{-D}.$$

Hence suppose the contrary holds; that is, for each i we have

$$a_i\geqq D. \tag{4.16}$$

In the notation of (4.9) there are two possibilities:

(1) $v_1,\ldots,v_k$ are mutually distinct;

(2) $v_i=v_j$ for some $i<j$.

We shall show both cases give contradictions, thus proving the theorem.

Suppose (1) holds so that $v_1,\ldots,v_k$ are mutually distinct. Then (4.14), (4.15), and (4.16) imply

$$v_i/p_i^2=\sigma(p_i^{a_i})/u_ip_i^2>2^{a_i-2}/u_i\geqq 2^{D-2}/u_i>1$$

so that

$$v_1 > 3 \text{ and } S \doteq \sum_{i=2}^{k} \frac{1}{v_i} < \sum_{i=2}^{k} \frac{1}{p_i^2} < \zeta(2) - 1 < \frac{2}{3}. \tag{4.17}$$

Let $v = \Pi v_i$. Then $v | \sigma(\Pi p_i^{a_i}) | \sigma(n) | cn$. But $(v, \Pi p_i) = 1$ and $c | \Pi p_i$, so $v | n$. Then $v \Pi p_i^{a_i} | n$. Now

$$\frac{\sigma(v)}{v} = \sum_{d|v} \frac{1}{d} \geqq 1 + \Sigma \frac{1}{v_i} = \frac{v_1 + 1}{v_1} + S \tag{4.18}$$

since $v_1, \ldots, v_k$ are distinct. Also [using (4.9)]

$$\begin{aligned} \Pi\left(\frac{\sigma(p_i^{a_i})}{p_i^{a_i}} \cdot \frac{p_i - 1}{p_i}\right) &= \Pi \frac{p_i^{a_i+1} - 1}{p_i^{a_i+1}} = \Pi\left(1 - \frac{1}{p_i^{a_i+1}}\right) \\ &= \Pi\left(1 - \frac{1}{1 + (p_i - 1) u_i v_i}\right) \\ &> \left(1 - \frac{1}{1 + v_1}\right) \prod_{i=2}^{k} \left(1 - \frac{1}{2v_i}\right) \\ &\geqq \frac{v_1}{v_1 + 1}\left(1 - \sum_{i=2}^{k} \frac{1}{2v_i}\right). \end{aligned} \tag{4.19}$$

Multiplying (4.18) and (4.19) and using (4.17) we have

$$\begin{aligned} 1 = \frac{\sigma(n)}{n} \cdot \Pi \frac{p_i - 1}{p_i} &\geqq \frac{\sigma(v)}{v} \cdot \Pi\left(\frac{\sigma(p_i^{a_i})}{p_i^{a_i}} \cdot \frac{p_i - 1}{p_i}\right) \\ &> \left(\frac{v_1 + 1}{v_1} + S\right) \cdot \frac{v_1}{v_1 + 1} \cdot \left(1 - \frac{1}{2} S\right) \\ &> \left(1 + \frac{3}{4} S\right)\left(1 - \frac{1}{2} S\right) = 1 + \frac{1}{4} S\left(1 - \frac{3}{2} S\right) > 1. \end{aligned}$$

This contradiction shows that case (1) is impossible.

We now assume case (2) holds; that is $v_i = v_j$ for some $i < j$. To simplify notation we shall let

$$p = p_i, q = p_j, a = a_i + 1, b = a_j + 1.$$

Then

$$p < q, a > D, b > D.$$

Let

$$\beta = \frac{u_j(p_j - 1)}{u_i(p_i - 1)} = \frac{\sigma(p_j^{a_j})(p_j - 1)}{\sigma(p_i^{a_i})(p_i - 1)} = \frac{q^b - 1}{p^a - 1}.$$

If $\beta = 1$, then $q^b - 1 = p^a - 1$, so that $q = p$, a contradiction. Hence (4.14) implies

$$1 < \max\{\beta, 1/\beta\} < p_k^{K + (k-1)(p_k - 1)}, \tag{4.20}$$

so that

$$0 < |\log \beta| < D/1000 p_k. \tag{4.21}$$

Note that also

$$\left|\beta-\frac{q^b}{p^a}\right|=\left|\frac{(q^b-1)-(p^a-1)}{p^a(p^a-1)}\right|=\frac{|\beta-1|}{p^a}. \tag{4.22}$$

From (4.21) we have

$$a\log p > D\log 2 > \left(\frac{2}{3}D+3\right)\log 2$$
$$>(2|\log\beta|+3)\log 2 > \log(2(\beta+1)/\beta)$$

so that $p^a > 2(\beta+1)/\beta$. Hence from (4.22) we have $|\beta-q^b/p^a| \leqq (\beta+1)/p^a < \beta/2$, so that

$$\beta/2 < \min\{\beta, q^b/p^a\} < \max\{\beta, q^b/p^a\} < 3\beta/2. \tag{4.23}$$

Hence we have

$$\Gamma \doteq |\log\beta - b\log q + a\log p| < (2/\beta)|\beta - q^b/p^a|$$
$$= 2|1-1/\beta|/p^a.$$

Then

$$T \doteq -\log\Gamma > -\log(2|1-1/\beta|) + a\log p$$
$$> -2 - |\log\beta| + a\log 2.$$

Hence

$$T > 0.69a \tag{4.24}$$

since $T \leqq 0.69a$ implies $a < (2+|\log\beta|)(0.003)^{-1} < D$, contradicting (4.21).

We also have

$$a \geqq b, \tag{4.25}$$

for if $a < b$, then by (4.21)

$$q^b/p^a > (1+1/p)^a > (1+1/p_k)^D > (1+1/p_k)^{2p_k\log(3\beta/2)}$$
$$\geqq (1+1/3)^{6\log(3\beta/2)} > 3\beta/2,$$

contradicting (4.23).

We use Theorem 2.1 for the inequality

$$0 < |\log\beta - b\log q + a\log p| = e^{-T} < e^{-0.69a}$$

where we may take $A = p_k^{K+(k-1)(p_k-1)}$ [cf. (4.20)], $H = a$ [cf. (4.25)], and $\delta = 0.69$ [cf. (4.24)]. The conclusion is that

$$a < 2^{1470}(0.69)^{-49}([K+(k-1)(p_k-1)]\log p_k)^{49} < D,$$

a contradiction. Hence we have a contradiction for case (2) as well as case (1), thus completing the proof of Theorem 4.2.

We now turn to the proof of Theorem 4.1:

Proof of Theorem 4.1. Suppose $\sigma(n)/n = \alpha$, n is primitive, and $\omega(n) = K$. Theorem 4.2 gives us effectively computable upper bounds for each of the prime factors of n.

Indeed if the prime factorization of n is $p_1^{a_1}\dots p_K^{a_K}$ where $p_1<\dots<p_K$, then Theorem 4.2 implies for $i=1,\dots,K$ that

$$p_i \leqq p(\alpha, K, i)$$

where $p(\alpha, K, i)$ is effectively computable. Hence to prove Theorem 4.1 all we need do is effectively compute upper bounds for the exponents $a_1,\dots,a_K$.

From the equation $\sigma(n)=\alpha n=cn/d$, Lemma 3.2 gives us for any i,j

$$\begin{aligned} v_{p_i}(a_j+1) &\leqq \tau(a_j+1)-1 \leqq \omega(\sigma(p_j^{a_j}))+1 \\ &\leqq \omega(\sigma(n))+1 \leqq \omega(c)+K+1\,. \end{aligned} \tag{4.26}$$

Then Lemma 3.1, (4.12), (4.13), and (4.26) give for any i

$$\begin{aligned} a_i = v_{p_i}(n) &\leqq v_{p_i}(d)+v_{p_i}(\sigma(n)) = v_{p_i}(d)+\sum_{j\neq i} v_{p_i}(\sigma(p_j^{a_j})) \\ &\leqq v_{p_i}(d)+\sum_{j\neq i} v_{p_i}(a_j+1)+\sum_{j\neq i}(p_i-1)\log p_j/\log p_i \\ &\leqq v_{p_i}(d)+(K-1)(\omega(c)+K+1)+(K-1)(p_K-1) \\ &\leqq \Omega(d)+(K-1)(\omega(c)+K+p(\alpha,K,K))\,. \end{aligned} \tag{4.27}$$

This completes the proof of Theorem 4.1.

§ 5. Odd Solutions of (1.1)

In this section we obtain the explicit upper bound (1.2) for the odd solutions of (1.1) with exactly K distinct prime factors. In view of (4.10), we note that the lengthy and difficult case $h(\Pi p_i)=\alpha$ in the proof of Theorem 4.2 is not necessary in the consideration of odd solutions of (1.1).

We denote by p', r', s' the functions defined in the statement of Theorem 4.2 corresponding to odd solutions of (1.1). In computing $r'(\alpha, K, k)$ we need only consider (4.1), (4.5), (4.7), and (4.8). It is not hard to show (by induction) that if $K \geqq k \geqq 2$, then

$$r'(\alpha, K, k) > \left(2cK \prod_{i=2}^{k} p'(\alpha, K, i)\right)^{-2^k}. \tag{5.1}$$

Hence another induction argument coupled with (4.2), (4.4), and (4.6) shows that if $K \geqq k \geqq 1$, then

$$p'(\alpha, K, k) < (2cK)^{2^{k(k+1)/2}}. \tag{5.2}$$

Now suppose $n=\prod_{k=1}^{K} p_k^{a_k}$ is an odd solution of (1.1) with the K distinct prime factors $p_1,\dots,p_K$. Then from (4.27) and (5.2) we have

$$\begin{aligned} a_k &\leqq \Omega(d)+(K-1)(\omega(c)+K+p'(\alpha,K,K)) \\ &< cK+K^2+(K-1)(2cK)^{2^{K(K+1)/2}} \\ &< K(2cK)^{2^{K(K+1)/2}}. \end{aligned}$$

Hence for $K \geqq 2$,

$$\begin{aligned} n = \Pi p_k^{a_k} &< p'(\alpha, K, K)^{K^2 \cdot (2cK)^{2^{K(K+1)/2}}} \\ &< (2cK)^{2^{K(K+1)/2} \cdot K^2 \cdot (2cK)^{2^{K(K+1)/2}}} \\ &< (2cK)^{(2cK)^{2^{K(K+1)/2+1}}} \\ &\leqq (2cK)^{(2cK)^{2^{K^2}}} . \end{aligned}$$

Also we note that this estimate holds for $K = 1$, since if $n = p^a$, then $c/d = \alpha = \sigma(p^a)/p^a$, so $n = p^a = d < c$.

§ 6. Anderson's Conjecture

In [2], Anderson conjectured that the set of rationals of the form $\sigma(n)/n$ is a recursive set. We prove:

Theorem 6.1. *Suppose either that the set of Mersenne primes is infinite or that there is an effectively computable upper bound for the set of Mersenne primes. Then for any K, the set $R(K)$ of rationals of the form $\sigma(n)/n$ where $\omega(n) \leqq K$ is a recursive set.*

Proof. Let α be a rational number. Let S_1 be the set of all primitive integers n such that $\sigma(n)/n = \alpha$ and $\omega(n) \leqq K$. Let S_2 be the set of all odd integers m such that $\sigma(m)/m = \alpha/2$ and $\omega(m) \leqq K-2$. Theorem 4.1 implies that there are effectively computable upper bounds for both S_1 and S_2. If $S_1 \neq \emptyset$, then $\alpha \in R(K)$. If $S_1 = S_2 = \emptyset$, then $\alpha \notin R(K)$. Suppose $S_1 = \emptyset$ and $S_2 \neq \emptyset$. If there are infinitely many Mersenne primes, then for each $m \in S_2$, there is an even perfect number e with $(e, m) = 1$, so that $\sigma(em)/em = \alpha$. Hence $\alpha \in R(K)$. Suppose there is an effectively computable upper bound for the set of Mersenne primes. Then for each $m \in S_2$ we can effectively decide whether or not there is an even perfect number e for which $(e, m) = 1$. If there exists such a pair e, m, then $\alpha \in R(K)$. If not, then $\alpha \notin R(K)$.

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