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SOME NEW RESULTS ON ODD PERFECT NUMBERS

G. G. DANDAPAT, J. L. HUNSUCKER AND
CARL POMERANCE

If m is a multiply perfect number ($\sigma(m) = tm$ for some integer t), we ask if there is a prime p with $m = p^a n$, $(p^a, n) = 1$, $\sigma(n) = p^a$, and $\sigma(p^a) = tn$. We prove that the only multiply perfect numbers with this property are the even perfect numbers and 672. Hence we settle a problem raised by Suryanarayana who asked if odd perfect numbers necessarily had such a prime factor. The methods of the proof allow us also to say something about odd solutions to the equation $\sigma(\sigma(n)) = 2n$.

1. Introduction. In this paper we answer a question on odd perfect numbers posed by Suryanarayana [17]. It is known that if m is an odd perfect number, then $m = p^a k^2$ where p is a prime, $p \nmid k$, and $p \equiv a \equiv 1 \pmod 4$. Suryanarayana asked if it necessarily followed that

$$\sigma(k^2) = p^a , \qquad \sigma(p^a) = 2k^2 . \tag{1}$$

Here, σ is the sum of the divisors function. We answer this question in the negative by showing that no odd perfect number satisfies (1).

We actually consider a more general question. If m is multiply perfect ($\sigma(m) = tm$ for some integer t), we say m has *property S* if there is a prime p with $m = p^a n$, $(p^a, n) = 1$, and the equations

$$\sigma(n) = p^a , \qquad \sigma(p^a) = tn \tag{2}$$

hold. Note that if n, p, a, t is a solution of (2) with p prime, then $1 = (p^a, \sigma(p^a)) = (p^a, n)$, so that $\sigma(p^a n) = tp^a n$; that is $p^a n$ is multiply perfect. Hence the multiply perfect numbers with property S are in one-to-one correspondence with the solutions of (2). We shall prove:

THEOREM 1. *If p is a prime, n, a, t are positive integers, and (2) holds, then either*

$$n = 21 , \quad p = 2 , \quad a = 5 , \quad t = 3 \tag{3}$$

or

$$n = 2^k , \quad p = 2^{k+1} - 1 , \quad a = 1 , \quad t = 2 . \tag{4}$$

COROLLARY. *If m is a multiply perfect number with property*

S, then $m = 672$ or m is an even perfect number. In particular, no odd perfect number has property S.

Write the odd perfect number $m = p^a k^2$ as a product of primes $p^a p_1^{2a_1} \cdots p_\nu^{2a_\nu}$. (Note that Pomerance [12] and Robbins [14] have shown that $\nu \geqq 6$.) Let $N(m)$ be the number of subscripts i for which there is a subscript j such that $(\sigma(p_i^{2a_i} p_j^{2a_j}), p_i p_j) > 1$. Then $0 \leqq N(m) \leqq \nu$. It is not difficult to see that Suryanarayana's equations (1) are equivalent to the odd perfect m satisfying $N(m) = 0$. Hence the above corollary implies $N(m) > 0$. We show however that $N(m)$ is not even close to 0, but more nearly ν.

THOREM 2. *If m is an odd perfect number, then*

$$\nu + 1 - [\log(\nu + 1)/\log 2] \leqq N(m) \leqq \nu . \tag{5}$$

Several authors (Kanold [8], Niederreiter [11], Suryanarayana [16], [18]) have considered the equation

$$\sigma(\sigma(n)) = 2n , \tag{6}$$

calling the solutions n super perfect. The even super perfects have been completely classified, but it is not known if any odd super perfects exist. The methods we develop to consider (1), (2), and (5) allow us also to get some results on odd solutions of (6). We shall prove:

THEOREM 3. *If n is an odd super perfect number, then neither n nor $\sigma(n)$ is a prime power and either n or $\sigma(n)$ is divisible by at least 3 distinct primes.*

Note that Suryanarayana [18] has already shown that n is not a prime power, but we give a new proof here for completeness. We (the second and third authors) have actually been able to prove much more than Theorem 3, but we do not give the details in this paper. (We have proved that if n is an odd super perfect number, then $n > 7 \cdot 10^{24}$, $\omega(n\sigma(n)) \geqq 5$, and $\omega(n) + \omega(\sigma(n)) \geqq 7$. Here $\omega(n)$ is the number of distinct prime factors of n.)

The main tool of this paper (Theorem A in §2) has the remarkable distinction of having been proved independently nine times.

In the research for this paper, the first author worked separately from the other authors.

2. **Preliminaries.** If x, y are integers, we shall write $x \| y$ if $x \mid y$ and $(x, y/x) = 1$. If p, q are distinct primes, we shall denote by $\mathrm{ord}_q(p)$ the exponent p belongs to mod q, that is, the smallest

natural number d for which $p^d \equiv 1 \pmod{q}$. We denote by $a_q(p)$ the integer e such that $q^e \| p^d - 1$, where $d = \text{ord}_q(p)$. Clearly $\text{ord}_q(p) \mid q-1$ and $a_q(p) \geqq 1$.

From Theorems 94 and 95 in Nagell [10] and the fact that $\sigma(p^x) = (p^{x+1} - 1)/(p - 1)$, we have:

LEMMA 1. *Suppose p, q are distinct primes with $q \neq 2$ and b, c are natural numbers. Then*

(i) *if $p \equiv 1 \pmod{q}$, then $q^b \| \sigma(p^c)$ if and only if $q^b \| c + 1$,*

(ii) *if $p \not\equiv 1 \pmod{q}$, then $q^b \| \sigma(p^c)$ if and only if $b \geqq a_q(p)$, $\text{ord}_q(p) \mid c + 1$, and $q^{b-a_q(p)} \| c + 1$.*

LEMMA 2. *Suppose p, q are distinct primes, x, y, b, c are natural numbers, $\sigma(q^x) = p^y$ and $q^b \| \sigma(p^c)$. Assume $q \neq 2$. Then*

(i) *if $p \equiv 1 \pmod{q}$, then $q^b \| c + 1$,*

(ii) *if $p \not\equiv 1 \pmod{q}$, then $\text{ord}_q(p) \mid c + 1$ and $q^{b-1} \| c + 1$.*

Proof. Now (i) follows from (i) of Lemma 1. Also (ii) will follow from (ii) of Lemma 1 provided we show $a_q(p) = 1$. Now $p^y = \sigma(q^x) = 1 + q + \cdots + q^x$, so that $p^y - 1 \equiv q \pmod{q^2}$. Then since $p \not\equiv 1 \pmod{q}$, we have $q \| (p^y - 1)/(p - 1) = \sigma(p^{y-1})$. Lemma 1 now implies $a_q(p) = 1$.

There is a well-known result about expressions of the form $(a^b - 1)/(a - 1)$ (see Bang [2], Zsigmondy [20], Sylvester [19], Birkhoff and Vandiver [3], Dickson [4], Kanold [7], Artin [1], Leopoldt [9], Richter [13]), which implies the following:

THEOREM A. *If p is a prime, x is a natural number, and $1 < d \mid x + 1$, then there is a prime $q \mid \sigma(p^x)$ with $\text{ord}_q(p) = d$, unless*

(i) *$p = 2$ and $d = 6$,*

(ii) *p is a Mersenne prime* (that is, of the form $2^k - 1$) *and $d = 2$.*

3. The main results. In this section we prove Theorems 1 and 2.

Proof of Theorem 1. We first consider the case $p = 2$. From the equation $\sigma(n) = 2^a$ and Theorem A, we see that n is a product of distinct Mersenne primes (cf. Sierpiński [15]); say $n = p_1 p_2 \cdots p_s$ where each $p_i = 2^{k_i} - 1$, k_i is prime and $k_1 < k_2 < \cdots < k_s$. Then $a = \sum k_i$. Now $tn = \sigma(2^a) = 2^{1+\Sigma k_i} - 1$. Hence for $1 \leqq j \leqq s$, we have $2^{k_j} - 1 \mid 2^{1+\Sigma k_i} - 1$, so that $k_j \mid \sum k_i$. Since the k_j are distinct primes, we have

$$(7) \qquad \prod_{i=1}^{s} k_i \Big| 1 + \sum_{i=1}^{s} k_i \;.$$

Then $s \geqq 2$. Now the expression $\prod k_i - 1 - \sum k_i$ increases separately in each of the s "variables" $k_1, k_2, \cdots, k_s$. If $s = 2$, $k_1 = 2$, $k_2 = 3$, we have $2 \cdot 3 \mid 1 + 2 + 3$. This gives the solution (3). If $s = 2$ and $k_2 \geqq 5$, then $k_1k_2 - 1 - k_1 - k_2 \geqq 2 \cdot 5 - 1 - 2 - 5 > 0$, so that (7) fails. Also if $s \geqq 3$, $\prod k_i - 1 - \sum k_i > 2^s - 1 - 2s > 0$, so again (7) fails.

We now consider the case $p > 2$. Since $\sigma(n) = p^a$ is odd, we have $n = 2^k p_1^{2a_1} \cdots p_r^{2a_r}$ where $k \geqq 0$, $r \geqq 0$, and $p_1, \cdots, p_r$ distinct odd primes. Suppose $r = 0$, so that $n = 2^k$. Then $\sigma(n) = 2^{k+1} - 1 = p^a$. Suppose $a > 1$. By Theorem A, there is a prime $q \mid \sigma(p^{2a-1})$ with $\mathrm{ord}_q(p) = 2a$. Then $q \mid (p^{2a} - 1)/(p^a - 1) = p^a + 1 = 2^{k+1}$, an impossibility since q is odd (cf. Gerono [6]). Hence $a = 1$ and we have solution (4). Thus we may assume $r \geqq 1$. Now for $1 \leqq i \leqq r$, we have $\sigma(p_i^{2a_i}) \mid p^a$ and $p_i^{2a_i} \mid \sigma(p^a)$. Lemma 2 then implies $p_i \mid a + 1$, so that $p_1p_2 \cdots p_r \mid a + 1$. Theorem A implies there is a prime $q \mid \sigma(p^a)$ with $\mathrm{ord}_q(p) = p_1p_2 \cdots p_r$. Then $q \neq 2, p_1, \cdots, p_r$, and since $q \mid tn$, we have $q \mid t$. Hence

$$\begin{aligned} p_1p_2 \cdots p_r < q \leqq t = \frac{\sigma(p^a)}{n} &= \frac{\sigma(p^a)}{p^a} \cdot \frac{\sigma(n)}{n} \\ &= \frac{p^{a+1} - 1}{p^a(p-1)} \cdot \frac{2^{k+1} - 1}{2^k} \cdot \prod \frac{p_i^{2a_i+1} - 1}{p_i^{2a_i}(p_i - 1)} \\ &< \frac{p}{p-1} \cdot 2 \cdot \prod \frac{p_i}{p_i - 1}\,, \end{aligned}$$

so that

$$1 < \frac{2p}{p-1} \cdot \prod \frac{1}{(p_i - 1)} \leqq \frac{2p}{(p-1)(p_1 - 1)} \leqq \frac{2 \cdot 3}{2 \cdot 4} < 1\,,$$

a contradiction.

Proof of Theorem 2. If i is such that $1 \leqq i \leqq \nu$ and $(\sigma(p_i^{2a_i} p_j^{2a_j}), p_ip_j) = 1$ for all j, $1 \leqq j \leqq \nu$, then $p_i^{2a_i} \mid \sigma(p^a)$ and $\sigma(p_i^{2a_i}) \mid p^a$. Let Ω be the set of such subscripts i, and let ω be the cardinality of Ω. Lemma 2 implies that $\prod_\Omega p_i \mid a + 1$. Since also $2 \mid a + 1$, we have at least $2^{\omega+1} - 1$ divisors d of $a + 1$ with $d > 1$. Since p is not a Mersenne prime (we have $p \equiv 1 \pmod 4$), Theorem A implies for each such d, there is a prime $r = r_d \mid \sigma(p^a)$ with $\mathrm{ord}_r(p) = d$. Then each r_d is odd, and since m is perfect, we have $r_d \in \{p_1, p_2, \cdots, p_\nu\}$. Hence $2^{\omega+1} - 1 \leqq \nu$, so that $\omega \leqq [\log(\nu + 1)/\log 2] - 1$.

4. Super perfect numbers.

LEMMA 3. *Let n be an odd super perfect number. Then*
(i) *n is a square,*
(ii) *$\sigma(n)$ is odd,*
(iii) *the prime factorization of $\sigma(n)$ is $p^a p_1^{2a_1} \cdots p_\nu^{2a_\nu}$ where* $p \equiv a \equiv 1 \pmod 4$ *and* $\nu \geqq 0$.

Proof. Kanold [8] proved (i) and (ii). Then $m = \sigma(n)$ is an odd integer for which $2 \| \sigma(m)$. Then such an odd integer must have the prime factorization indicated in (iii) (cf. Euler [5]).

Proof of Theorem 3. Suppose $\sigma(n)$ is the prime power p^a. Then $\sigma(p^a) = \sigma(\sigma(n)) = 2n$, so that Theorem 1 implies $p^a n$ is even, contradicting Lemma 3.

Suppose n is the prime power q^b. Then, in the notation of Lemma 3, we have just proved that $\nu \geqq 1$, so that for $1 \leqq i \leqq \nu$ we have $p_i^{2a_i} \mid \sigma(q^b)$ and $\sigma(p_i^{2a_i}) \mid q^b$. Say $r = \max\{p_1, p_2, \cdots, p_\nu\}$. Now Lemma 2 implies either $r^2 \mid b+1$ or $r \cdot \mathrm{ord}_r(q) \mid b+1$ in which case $\mathrm{ord}_r(q) > 1$. In the first case $b+1$ has the 2 divisors r and r^2 which are multiples of r. In the second case, $b+1$ has the 2 divisors r and $r \cdot \mathrm{ord}_r(q)$ which are multiples of r. Since q is odd, in either case Theorem A implies there are 2 distinct primes dividing $\sigma(q^b)$ which are $1 \pmod r$. This contradicts (iii) of Lemma 3 and the choice of r.

Suppose both n and $\sigma(n)$ are divisible by precisely 2 distinct primes. Now if $(n, \sigma(n)) = 1$, then $n\sigma(n)$ is divisible by precisely 4 distinct primes and $\sigma(n\sigma(n)) = \sigma(n)\sigma(\sigma(n)) = 2n\sigma(n)$. Then Lemma 3 implies $n\sigma(n)$ is an odd perfect number. This contradicts the previously stated result ([12], [14]) that every odd perfect number is divisible by at least 7 distinct primes. Hence $(n, \sigma(n)) > 1$. Hence from Lemma 3 we have the prime factorizations

$$n = q^{2b} r^{2c}$$
$$\sigma(n) = q^\alpha s^\beta .$$

Now $\sigma(q^{2b}) \mid s^\beta$ and since $n \mid \sigma(\sigma(n))$, we have $q^{2b} \mid \sigma(s^\beta)$. Then, as in the above paragraph, there are at least 2 distinct primes dividing $\sigma(s^\beta)$ which are $1 \pmod q$. This contradicts $\sigma(s^\beta) \mid 2n$.

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INDIAN INSTITUTE OF TECHNOLOGY
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UNIVERSITY OF GEORGIA