ON THE PROBLEM OF UNIQUENESS FOR THE MAXIMUM STIRLING NUMBER(S) OF THE SECOND KIND

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Abstract

Say that an integer n is *exceptional* if the maximum Stirling number of the second kind $S(n,k)$ occurs for two (of necessity consecutive) values of k. We prove that the number of exceptional integers less than or equal to x is $O(x^{3/5+\epsilon})$, for any $\epsilon > 0$.

1. Introduction

Let $S(n,k)$ be the Stirling number of the second kind, that is, the number of partitions of an n-set into k non empty, pairwise disjoint blocks. (Detailed definitions appear in the next section.) Using the initial value $S(0,k) = \delta_{0\,k}$ and the recursion

$$S(n+1,k) \;=\; kS(n,k) \;+\; S(n,k-1) \tag{1}$$

one may show by induction on n that

$$S(n,k)^2 \;\geq\; (1+\frac{3}{k})\,S(n,k-1)\,S(n,k+1), \qquad 1 \leq k \leq n. \tag{2}$$

It follows that the ratio $S(n,k+1)/S(n,k)$ is strictly decreasing, and so there is either a unique maximum Stirling number

$$S(n,k) \;<\; S(n,K_n), \quad \text{for all } k \neq K_n$$

or else there are two consecutive peaks

$$S(n,k) \;<\; S(n,K_n) \;=\; S(n,K_n+1), \quad \text{for all } k \notin \{K_n, K_n+1\}.$$

Define the *exceptional set* E to be those n for which the second alternative holds. Based on computation through $n = 10^6$ reported in the final section, it is possible that $E = \{2\}$. Let $E(x)$ denote the associated counting function

$$E(x) = \#\{n : n \leq x \text{ and } n \in E\}.$$

The purpose of this paper is to prove

Theorem 1. For any $\epsilon > 0$,

$$E(x) = O(x^{3/5+\epsilon}).$$

Our proof of this theorem depends on the fact that, when $n \in E$, the quantity e^r, where r is the unique real solution of the equation $re^r = n$, must be unusually close to an integer plus $1/2$. (See equation (5) in Section 3.) Starting from (5) and using only elementary arguments, we will prove in Section 4 a result slightly weaker than Theorem 1, namely with the exponent $3/5$ replaced by $2/3$. Then, in Section 5, we will prove Theorem 1 by invoking recent work of Huxley [9] on counting integer points near curves. In Section 6, we give a heuristic argument for why E should be a finite set. Finally, in Section 7, we report on the computation and supporting lemma that proves $E \cap (1, 10^6] = \emptyset$.

2. Definitions and Background

A *partition* of the set $[n] = \{1, 2, \ldots, n\}$ is a collection of non empty pairwise disjoint subsets of $[n]$, called *blocks*, whose union equals $[n]$. For example, $\{\{1, 4\}, \{2, 3, 5, 7\}, \{6\}\}$ is a partition of $[7]$ into 3 blocks. The Stirling number of the second kind, $S(n, k)$, is the number of partitions of $[n]$ into k blocks. Every partition of $[n+1]$ into k blocks can be obtained either by adjoining $\{n+1\}$ as a singleton block to an existing partition of $[n]$ into $k-1$ blocks, or by adding the element $n+1$ to one of the blocks of an existing partition of $[n]$ into k blocks. This construction proves the recursion (1). Here is a table of the first few rows of the Stirling numbers of the second kind:

$n\backslash k$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

As explained in the Introduction, for each $n \geq 1$ there is a unique integer K_n satisfying

$$S(n,1) \; < \; \cdots \; < \; S(n,K_n) \; \geq \; S(n,K_n+1) \; > \; \cdots \; > \; S(n,n). \tag{3}$$

In words, K_n is the location of the maximum Stirling number of the second kind, with the proviso that should there be two consecutive maxima, K_n is the location of the "leftmore." The *exceptional set* E consists of n such that $S(n,K_n) = S(n,K_n+1)$, and $E(x)$ is the number of $n \leq x$ belonging to E.

There is a vast literature on the Stirling numbers to which many people have contributed, and many properties have been independently rediscovered. Harper's [7] contributions are particulary noteworthy. He shows that the polynomials $\sum_k S(n,k)x^k$ have only real roots, a property called *total positivity*, which is stronger than log concavity. He articulates the unique-or-double peak property (3), and proves the asymptotic relation $K_n \sim n/\log n$, (His formula contains a superflous factor e which was later corrected.) The asymptotic formula was obtained by others, for example [18]. Citing Harper's work, Lieb [13] derives an inequality similar to (2), based on the general *Newton Inequality* for coefficients of polynomials whose roots are all real and negative. The very nice fact that K_{n+1} equals either K_n or K_n+1 appears in [4] and [16]. Using (1) and (2), it can be shown that a necessary condition for $n \in E$ is $K_{n+1} = K_n + 1$. Thus, the growth condition $K_{n+1} - K_n \in \{0,1\}$ plus the asymptotic relation $K_n \sim n/\log n$ together imply that $E(x) = O(x/\log x)$, as first pointed out by Wegner [19]. The latter paper of Wegner makes the explicit conjecture that $E = \{2\}$. Prior to the general adoption of more powerful analytic tools, in a series of papers [1, 6, 10, 11, 12] the authors Bach, Harborth, and Kanold employ clever elementary arguments to prove many interesting, sharp inequalities about K_n.

The fact that the signless Stirling numbers of the first kind do indeed have always a unique maximum is due to Erdős [5].

The status of the "duplicate maximum" problem has been misstated in the literature more than once. A source of misunderstanding might be the one line abstract, perpetuated in the Mathematical Reviews, of [4] which states, "For fixed n, Stirling numbers of the second kind, $S(n,r)$, have a single maximum." Reading the paper, one sees clearly that the intended meaning is precisely (3); but certainly the statement can be easily misconstrued when read in isolation.

Canfield [2] and Menon [14] independently showed that K_n is always equal to $\lfloor \kappa(n) \rfloor$ or $\lceil \kappa(n) \rceil$, where $\kappa(n)$ is a certain transcendentally defined function. It will follow from what we say in Section 3 that for sufficiently large n a simpler definition of $\kappa(n)$ also satisfies the latter theorem, namely $\kappa(n) = e^r - 1$, where $re^r = n$. Throughout the paper, we shall always use $r(x)$ for the implicitly defined function

$$r(x)e^{r(x)} \;\; = \;\; x,$$

and the symbol r, with no argument, denotes $r(n)$. For $1 \leq n \leq 1200$ there is no exception to the relation

$$K_n \quad \in \quad \{\lfloor e^r - 1 \rfloor, \lceil e^r - 1 \rceil\}, \tag{4}$$

although it has been proven true only for n sufficiently large.

3. Asymptotics of the Stirling Numbers $S(n,k)$

We will neglect polylog factors in our estimates, and so it is convenient to define

$$F_1(x) \quad = \quad O_*(F_2(x))$$

to mean that for a sufficiently large constant C we have

$$|F_1(x)| \quad \leq \quad C(\log x)^C F_2(x), \quad \text{ for } x \geq C.$$

This given, we may state the lemma that will be of central importance.

Lemma 1. For all sufficiently large $n \in E$ we have

$$e^r \quad = \quad \lfloor e^r \rfloor \ + \ \frac{1}{2} \ + \ \frac{1/2}{1+r} \ + \ O_*(n^{-1}), \tag{5}$$

where as usual $re^r = n$.

Proof. The exponential generating function in the letter n for $S(n,k)$ is [3]

$$\sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!} \quad = \quad \frac{(e^x-1)^k}{k!}.$$

The Cauchy integral formula thus asserts

$$\frac{S(n,k)}{n!} \quad = \quad \frac{1}{2\pi i k!}\oint_{|z|=R} \frac{(e^z-1)^k}{z^{n+1}}dz,$$

for any $R > 0$. If we take the radius R of the circle of integration to be the quantity r, and restrict attention to integers k which satisfy the relations

$$e^r - 1 \quad = \quad k + \theta, \qquad \theta = O(1),$$

while making estimates such as those found in [15], we arrive at

$$S(n,k) \quad = \quad \frac{(e^r-1)^k}{k!}\frac{n!}{r^n}\left(2\pi k B\right)^{-1/2}\left(1 \ - \ \frac{6r^2\theta^2 + 6r\theta + 1}{12re^r} \ + \ O_*(n^{-2})\right),$$

where

$$B \quad = \quad B(r) \quad = \quad \frac{re^{2r} - (r^2+r)e^r}{(e^r-1)^2}$$

depends on r only.

This is very similar to the formula (1) in [2], although the latter was unnecessarily conservative in the error estimate. Now, with k and θ as above, we find

$$\frac{S(n,k+1)}{S(n,k)} \quad = \quad 1 \; + \; \frac{(r+1)\theta - \frac{1}{2}r - 1}{e^r} \; + \; O_*(n^{-2}).$$

It is this equation which gives us the assertion (4), for all n large, mentioned earlier, and by setting the right side equal to 1, we obtain the lemma.

Remark. The asymptotic formula for $S(n,k)$, and the more detailed one appearing in the proof of Lemma 2 in Section 6, are obtained by using the circle method. We do not include any details about how to use this method, which is a very standard and widely used technique for obtaining asymptotic estimates of the coefficients of analytic functions. The reader for whom this is a new topic should study [20, Section 4.5] before moving on to other papers. A very good account of the circle method particularly useful for asymptotic enumeration is [8]. The paper of Moser and Wyman [15] contains a lot of useful information about the particular case of the Stirling numbers. Another good source for this topic is [17].

4. The Elementary Proof

Our goal in this section is to prove that for any $\epsilon > 0$

$$E(x) \quad = \quad O(x^{2/3+\epsilon}).$$

Let $\epsilon > 0$ be given. It suffices to show that for all sufficiently large X

$$\left| [X, X + X^{1/3-\epsilon/2}] \; \cap \; E \right| \quad \le \quad X^{\epsilon/2}. \tag{6}$$

If (6) fails, then we have infinitely many n such that $n, n+\ell_1, n+\ell_2 \in E$ with $0 < \ell_1 < \ell_2 \le n^{1/3-\epsilon/2}$ and $\ell_2 - \ell_1 > n^{\epsilon/4}$. For each such n, we have r with $re^r = n$, and also r_i with $r_i e^{r_i} = n + \ell_i$. Note that

$$\log x \; - \; \log\log x \quad \le \quad r(x) \quad \le \quad \log x,$$

whence

$$r_i \quad \sim \quad r.$$

Since $r(x)e^{r(x)} = x$, it follows that

$$e^{r_i} \quad \sim \quad e^r.$$

By Taylor's theorem and the facts that

$$\frac{d}{dx}e^{r(x)} \;=\; \frac{1}{r(x)+1}, \qquad \frac{d^2}{dx^2}e^{r(x)} \;=\; \frac{-1}{(r(x)+1)^3 e^{r(x)}}, \qquad \frac{d^3}{dx^3}e^{r(x)} \;=\; \frac{r(x)+4}{(r(x)+1)^5 e^{2r(x)}},$$

we have

$$\begin{aligned} e^{r_i} \;&=\; e^r \;+\; \frac{\ell_i}{r+1} \;-\; \frac{\ell_i^2}{2(r+1)^3 e^r} \;+\; O_*(\ell_i^3 n^{-2}), \\ \frac{1}{r_i+1} \;&=\; \frac{1}{r+1} \;-\; \frac{\ell_i}{(r+1)^3 e^r} \;+\; O_*(\ell_i^2 n^{-2}). \end{aligned}$$

Thus,

$$(\ell_1-\ell_2)e^r \;+\; \ell_2 e^{r_1} \;-\; \ell_1 e^{r_2} \;=\; -\frac{\ell_2\ell_1^2-\ell_1\ell_2^2}{2(r+1)^3 e^r} \;+\; O_*\Big(\frac{\ell_2\ell_1^3+\ell_1\ell_2^3}{n^2}\Big).$$

Similarly,

$$\frac{\ell_1-\ell_2}{1+r} \;+\; \frac{\ell_2}{1+r_1} \;-\; \frac{\ell_1}{1+r_2} \;=\; O_*\Big(\frac{\ell_2\ell_1^2+\ell_1\ell_2^2}{n^2}\Big).$$

Let us refer to the assertions of Lemma 1, namely,

$$e^{r_i} \;=\; m_i \;+\; 1/2 \;+\; \frac{1/2}{r_i+1} \;+\; O_*(n^{-1}),$$

as equation i, with $0 \le i \le 2$, taking $r_0 = r$. If we form $(\ell_1 - \ell_2)$ times equation 0 plus ℓ_2 times equation 1 minus ℓ_1 times equation 2, and substitute the above expansions, we find

$$-\frac{\ell_2\ell_1^2-\ell_1\ell_2^2}{2(r+1)^3 e^r} \;+\; O_*\Big(\frac{\ell_2\ell_1^3+\ell_1\ell_2^3}{n^2}\Big) \;=\; \text{INTEGER} \;+\; O_*\Big(\frac{\ell_2\ell_1^2+\ell_1\ell_2^2}{n^2}\Big) \;+\; O_*\Big(\frac{\ell_2}{n}\Big).$$

In the previous equation, every term except the one labeled "INTEGER" goes to 0 as $n \to \infty$; thus, for all sufficiently large n that term itself must be 0. Dividing through by $\ell_1\ell_2$ and collecting big-oh's,

$$\frac{\ell_2-\ell_1}{2(r+1)^3 e^r} \;=\; O_*\Big(\frac{\ell_1^2+\ell_2^2}{n^2}\Big) \;+\; O_*\Big(\frac{1}{\ell_1 n}\Big).$$

Since, however, $\ell_2 - \ell_1 \ge n^{\epsilon/4}$, this last equality is impossible. Our initial assumption that (6) does not hold is contradicted, and the proof is complete.

5. The Proof of Theorem 1

The theorem due to Huxley which we shall apply, [9, (1.7)], bounds the number of integer pairs (n, m) which satisfy $|m - f(n)| \le \delta$ for $n \in [X, 2X]$. We shall apply this result to the function

$$f(x) \;=\; e^{r(x)} \;-\; 1/2 \;-\; \frac{1/2}{1+r(x)},$$

with $\delta = X^{\epsilon-1}$. With these choices, by Lemma 1, for X sufficiently large, we include all members of $E \cap [X, 2X]$ in the count.

The hypotheses required of $f(x)$ are that there be numbers $C \geq 1$, $\Delta < 1$ such that

$$C\Delta \leq 1$$

$$\frac{\Delta}{C} \leq |f''(x)| \leq C\Delta, \quad x \in [X, 2X]$$

and

$$|f^{(3)}(x)| \leq \frac{C\Delta}{X}, \quad x \in [X, 2X].$$

The conclusion of Huxley's theorem is that the number of integer pairs (n, m) is no greater than an unspecified constant times

$$1 + \frac{1}{b}\sqrt{\frac{C\delta}{\Delta}} + C^2\delta X + \sum_{i=1}^{4}(C\Delta)^{a_i}X^{b_i}(\log X - \log 2C)^{c_i}\delta^{d_i},$$

where b is the least positive integer such that for some $x \in [X, 2X]$ we have $bf'(x)$ within distance δ of an integer, and the exponents (a_i, b_i, c_i, d_i) in the sum assume the four values $(\frac{2}{5}, 1, \frac{1}{10}, 0)$, $(\frac{1}{5}, \frac{4}{5}, \frac{1}{10}, 0)$, $(\frac{2}{7}, 1, \frac{1}{7}, \frac{1}{7})$, and $(\frac{1}{7}, \frac{6}{7}, \frac{1}{7}, \frac{1}{7})$.

If we take $\Delta = X^{-1}$, then the quantity C satisfying the hypotheses of Huxley's Theorem may be taken as $O(X^\epsilon)$, and we obtain the result that between X and $2X$ there are $O(X^{3/5+\epsilon})$ elements of E. This estimate suffices to prove the Theorem. (By being a bit more careful, one may use Huxley's Theorem to show that the number of members of E up to X is at most $X^{3/5}(\log X)^{\mathrm{O}(1)}$.)

6. A Heuristic

In this section we give a strengthening of Lemma 1 that leads to a heuristic argument that the set E is finite. Note that already the estimate of Lemma 1 heuristically supports the conclusion that $E(x) = O(x^\epsilon)$, and with more care, $E(x) \leq (\log x)^{\mathrm{O}(1)}$. To push this heuristic further we need a more precise version of Lemma 1.

Lemma 2. For $n \in E$ and $re^r = n$, we have

$$e^r = \lfloor e^r \rfloor + \frac{1}{2} + \frac{1/2}{1+r} + \frac{A_r}{e^r} + O_*\left(n^{-2}\right),$$

where A_r is a rational function in r with rational coefficients.

Proof. With the same meaning for k, θ, B as in the proof of Lemma 1, it is possible to show that uniformly for $|\theta| = O(1)$, we have

$$S(n,k) \;=\; \frac{(e^r-1)^k}{k!}\,\frac{n!}{r^n}\,(2\pi kB)^{-1/2}\Big(1 \;-\; \frac{F_1}{e^r} \;+\; \frac{F_2}{e^{2r}} \;+\; O\big(\frac{r^3}{e^{3r}}\big)\Big),$$

where

$$\begin{aligned} F_1(\theta) &= \frac{1+6r\theta+6r^2\theta^2}{12r}, \\ F_2(\theta) &= \frac{1}{288r^2}\Big(r^4(-36-144\theta-144\theta^2+36\theta^4) \\ &\quad + r^3(96+144\theta-288\theta^2-24\theta^3) \;+\; r^2(-144\theta-24\theta^2) \;+\; 12r\theta \;+\; 1\Big). \end{aligned}$$

Returning to the proof, it follows from the above formula that

$$\frac{S(n,k+1)}{S(n,k)} \;=\; \frac{e^r-1}{k+1}\Big(\frac{k}{k+1}\Big)^{1/2}\Big(\frac{1-F_1(\theta-1)e^{-r}+F_2(\theta-1)e^{-2r}}{1-F_1(\theta)e^{-r}+F_2(\theta)e^{-2r}} + O_*\left(n^{-3}\right)\Big). \qquad (7)$$

Suppose now that $n \in E$, so that $S(n,k+1)/S(n,k) = 1$. Write

$$\theta \;=\; u + \frac{1}{2} + \frac{1/2}{r+1},$$

so that Lemma 1 implies that $u = O_*(n^{-1})$. Hence, if $g(x,y) \in \mathbf{Q}[x,y]$, then

$$g(r,\theta) \;=\; g\Big(r, \frac{1}{2} + \frac{1/2}{r+1}\Big) + O_*\left(n^{-1}\right).$$

Thus,

$$\frac{e^r-1}{k+1} \;=\; \frac{e^r-1}{e^r-\theta} \;=\; 1 + \frac{\theta-1}{e^r} + \frac{\theta^2-\theta}{e^{2r}} + O_*\left(n^{-3}\right) \;=\; 1 + \frac{\theta-1}{e^r} + \frac{a_r}{e^{2r}} + O_*\left(n^{-3}\right),$$

where a_r is a rational function of r with rational coefficients. Also

$$\left(\frac{k}{k+1}\right)^{1/2} \;=\; 1 - \frac{1/2}{e^r} + \frac{1/8-\theta/2}{e^{2r}} + O_*\left(n^{-3}\right) \;=\; 1 - \frac{1/2}{e^r} + \frac{b_r}{e^{2r}} + O_*\left(n^{-3}\right),$$

where again, b_r is in $\mathbf{Q}(r)$. And

$$\begin{aligned} &\frac{1-F_1(\theta-1)e^{-r}+F_2(\theta-1)e^{-2r}}{1-F_1(\theta)e^{-r}+F_2(\theta)e^{-2r}} \;=\; \\ &1 - \frac{F_1(\theta-1)-F_1(\theta)}{e^r} + \frac{F_2(\theta-1)-F_2(\theta)+(F_1(\theta-1)-F_1(\theta))F_1(\theta)}{e^{2r}} + O_*\left(n^{-3}\right) \\ &= \; 1 + \frac{1/2-r/2+r\theta}{e^r} + \frac{c_r}{e^{2r}} + O_*\left(n^{-3}\right), \end{aligned}$$

where $c_r \in \mathbf{Q}(r)$.

Thus (7) and the above estimates imply that

$$1 \quad = \quad \left(1+\frac{\theta-1}{e^r}+\frac{a_r}{e^{2r}}\right)\left(1-\frac{1/2}{e^r}+\frac{b_r}{e^{2r}}\right)\left(1+\frac{1/2-r/2+r\theta}{e^r}+\frac{c_r}{e^{2r}}\right)+O_*\left(n^{-3}\right).$$

Subtracting 1 from both sides and multiplying by e^r, we get

$$\begin{aligned}
&(r+1)\theta-\tfrac{1}{2}-\tfrac{1}{2}(r+1)\\
&\quad= \; -e^{-r}\left(a_r+b_r+c_r-\tfrac{1}{2}(\theta-1)+(\theta-\tfrac{3}{2})(\tfrac{1}{2}-\tfrac{1}{2}r+r\theta)\right) \; + \; O_*\left(n^{-2}\right)\\
&\quad= \; d_r e^{-r} \; + \; O_*\left(n^{-2}\right),
\end{aligned}$$

where $d_r \in \mathbf{Q}(r)$. Thus, we have Lemma 2.

We now give a heuristic argument, based on Lemma 2, that the set E is finite. With $re^r = x$, let

$$g(x) \quad = \quad e^r-\frac{1}{2}-\frac{1/2}{r+1}-\frac{A_r}{e^r} \quad = \quad \frac{x}{r}-\frac{1}{2}-\frac{1/2}{r+1}-\frac{rA_r}{x}.$$

The function $g(x)$ is smooth with

$$g'(x) \;\sim\; \frac{1}{\log x}, \qquad g''(x) \;\sim\; \frac{-1}{x\log^2 x}, \qquad g^{(3)}(x) \;\sim\; \frac{1}{x^2\log^2 x}.$$

There is no reason to believe that $g(n)$ has a prediliction to be close to an integer over any other transcendental number. But Lemma 2 implies that for $n \in E$, we have $\|g(n)\| = O_*(n^{-2})$, where $\|\ \|$ denotes the distance to the nearest integer. Heuristically, the number of such integers n is $\sum O_*(n^{-2}) = O(1)$. (One might view the expression $O_*(n^{-2})$ as an upper bound for the "probability" that $n \in E$, and the sum of these probabilities is $O(1)$.)

7. Numerics

To verify that $E \cap (1, 10^6] = \emptyset$, we wrote a program to compute $S(n,k) \bmod 2^{31}-1$. We computed all such residues for $2 \le n \le 10^6$ and $2 \le k \le \min\{87890,\ n\}$, finding 33 pairs (n,k) satisfying the conditions:

$$\begin{aligned}
&2 \;\le\; n \;\le\; 10^6\\
&2 \;\le\; k \;<\; \min\{87890,\ 2n/\log(n),\ n\}\\
&S(n,k) = S(n,k+1) \bmod 2^{31}-1.
\end{aligned}$$

We may impose the stated bounds on k for these reasons: (1) by Lemma 3, stated and proven below, $K_n < 2n/\log(n)$ for $n \ge 151$; (2) an independent computation of exact values of $S(n,k)$, using `maple`, had already shown $E \cap (1, 1200] = \emptyset$; (3) $S(10^6, 87848) > S(10^6, 87890)$.

The third of these facts was established by making rigorous numerical estimates, with considerable help from `maple`. The basis for these estimates is the pair of inequalities

$$\frac{k^n}{k!}\sum_{j=0}^{\mathcal{O}}\binom{k}{j}(-1)^j(1-j/k)^n \quad\leq\quad S(n,k) \quad\leq\quad \frac{k^n}{k!}\sum_{j=0}^{\mathcal{E}}\binom{k}{j}(-1)^j(1-j/k)^n \tag{8}$$

for any positive odd integer $\mathcal{O}$ and nonnegative even integer $\mathcal{E}$. These are the Bonferroni inequalities ([3], Section 4.7). We used $\mathcal{O}=5$ and $\mathcal{E}=4$ to prove

$$\log S(10^6, 87848) \quad > \quad 10\,471\,198$$
$$\log S(10^6, 87890) \quad < \quad 10\,471\,197.992$$

Later, by taking $\mathcal{E}=10$ and $\mathcal{O}=11$ we were able to show conclusively that

$$K_{10^6} \quad = \quad 87846.$$

For anyone wishing to duplicate the computation, we provide these checkpoints:

- the first of the 33 pairs is $(n,k)=(124322, 16581)$
- the last of the 33 pairs is $(n,k)=(965756, 12911)$
- $S(10^6, 87890) = 1111899618 \bmod 2^{31}-1$
- $S(124322, 16581) = 1636672468 \bmod 2^{31}-1$
- $S(965756, 12911) = 897942184 \bmod 2^{31}-1$

The program was modified to compute $S(n,k) \bmod 2^{19}-1$, and run a second time. This second modulus was able to distinguish 31 of the pairs found in the first run; for example,

$$S(124322, 16581) = 31493 \bmod 2^{19}-1 \quad\text{and}\quad S(124322, 16582) = 504717 \bmod 2^{19}-1.$$

However, all four of the numbers $S(n,k)$ for $n=526314, k=51889, 51890$ and $n=559358, k=52358, 52359$ are 0 mod $2^{19}-1$. To distinguish among these a further calculation was needed. Note that the bounds given in equation (8) are in fact equalities if $\mathcal{E}$, or as appropriate $\mathcal{O}$, is equal to k. For a prime $p>k$ this provides a way to compute $S(n,k) \bmod p$ directly, without computing any other Stirling numbers in the process. This identity shows, as shown in [19, (4.1)], that $S(n,k)\equiv S(A,k) \bmod p$ for prime $p>k$ and $n\equiv A\not\equiv 0 \bmod (p-1)$. For the first few primes p larger than k, we have $0<A<k$, so for these primes $S(n,k)$ is congruent to 0. The first prime larger than 51889 for which $S(526314, 51889)$ is not congruent to 0 is $p=52639$. We have

$$S(526314, 51889) = 4890 \bmod 52639, \quad\text{and}\quad S(526314, 51890) = 43718 \bmod 52639.$$

In a similar manner, the pair for $n=559358$ is distinguished by the prime $p=55949$. In this way, then, we prove there are no duplicate maxima for $1<n\leq 10^6$.

We now conclude with the above referenced lemma.

Lemma 3. For all integers $n \geq 151$ we have $K_n < 2n/\log n$.

Proof. For any positive integer k with $1 \leq k \leq n$, we have

$$\frac{k^n}{k!} - \frac{(k-1)^n}{(k-1)!} \leq S(n,k) \leq \frac{k^n}{k!}. \tag{9}$$

These inequalities are the case $\mathcal{E} = 0$ and $\mathcal{O} = 1$ of equation (8). We include a from-scratch proof since it is not difficult. Indeed, the number of assignments of the integers $1, \ldots, n$ into k labeled boxes with no box remaining empty is at most k^n, and each set partition of $[n]$ corresponds to $k!$ such assignments. Thus, we have the upper bound in (9). Further, the number of assignments without the restriction that no box remain empty is exactly k^n, and the number of assignments where box i remains empty is exactly $(k-1)^n$. Thus, the number of assignments with no box remaining empty is at least $k^n - k(k-1)^n$. From this, the lower bound in (9) follows easily.

We now let $k = \lfloor n/\log n \rfloor$. We will show that for $n \geq 151$,

$$\frac{(2k)^n}{(2k)!} < \frac{k^n}{k!} - \frac{(k-1)^n}{(k-1)!}. \tag{10}$$

Note that (9) and (10) show that $S(n,k) > S(n,2k)$, and so from (3), we must have $K_n < 2k$. To see (10), let

$$\alpha = \frac{(2k)^n/(2k)!}{k^n/k!}, \qquad \beta = \frac{(k-1)^n/(k-1)!}{k^n/k!}.$$

We will show that for $n \geq 151$ we have $\alpha, \beta < 1/2$, so that (10) follows for these values of n.

We have

$$\begin{aligned} \beta = k(1-1/k)^n \leq ke^{-n/k} &= \lfloor n/\log n \rfloor e^{-n/\lfloor n/\log n \rfloor} \\ &\leq (n/\log n)e^{-\log n} = 1/\log n. \end{aligned}$$

Thus, for $n \geq 151$, we have $\beta \leq 1/\log 151 < 1/5$. The estimation for α is a little more difficult. We have

$$\alpha^{-1} = \frac{(2k)!}{k!}2^{-n} = k!\binom{2k}{k}2^{-n}.$$

Using the inequalities $k! > (k/e)^k$, $\binom{2k}{k} \geq 2^{2k}/(2k)$, which are both easy to see, we have

$$\alpha^{-1} > k^{k-1}e^{-k}2^{2k-1-n},$$

so that

$$\log(\alpha^{-1}) > (k-1)(\log k + \log 4 - 1) - ((n-1)\log 2 + 1).$$

An elementary check shows that this last expression exceeds 1 for all integers $n \geq 151$, so that $\alpha < 1/e$ in this range. This completes the proof of (10) and the lemma.

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