714 AND 715

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On April 8, 1974, in Atlanta, Georgia, Henry Aaron hit his 715th major league homerun, thus eclipsing the previous mark of 714 long held by Babe Ruth. This event received so much advance publicity that the numbers 714 and 715 were on millions of lips. Questions like "When do you think he'll get 715?" were perfectly understood, even with no mention made of Aaron, Ruth, or homerun.

In all of the hub-bub it appears certain interesting properties of 714 and 715 were overlooked. Indeed we first note that

$$714 \cdot 715 = 510510 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = P_7$$

where P_k denotes the product of the first k primes. Is this really unusual? That is, are there other pairs of consecutive numbers whose product is P_k for some k? We readily see that $1 \cdot 2 = P_1$, $2 \cdot 3 = P_2$, $5 \cdot 6 = P_3$, $14 \cdot 15 = P_4$. Putting the problem to the CDC 6400 at the University of Georgia (using 5 minutes of computer time), we found that the only P_k which can be expressed as the product of two consecutive numbers in the range $1 \leqslant k \leqslant 3049$ are P_1, P_2, P_3, P_4, and P_7. Hence if there is any other pair of consecutive integers whose product is a P_k, then these integers exceed 10^{6021}. Great as Henry Aaron is, we believe he will never again hit two consecutive homeruns whose numbers have their product equal to some P_k. However, on April 26, 1974, Henry Aaron did hit his 15th grand slam homerun, breaking the old National League mark of 14, and, of course, $14 \cdot 15 = P_4$.

Conjecture: The largest pair of consecutive integers whose product is also the product of the first k primes for some k is 714 and 715.

If the unique prime factorization of an integer n is $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, we let $S(n) = a_1 p_1 + a_2 p_2 + \cdots + a_k p_k$. It follows that for any natural numbers m, n we have $S(mn) = S(m) + S(n)$; that is, S is totally additive. A student at the University of Georgia, Jeremy Jordan, noted that

$$S(714) = S(715).$$

We shall call n an *Aaron* number if $S(n) = S(n+1)$. In the table we have the results of a computer search for all of the Aaron numbers not exceeding 20000.

The numerical data suggest that Aaron numbers are rare. We suspect they have density 0, but we cannot prove this. We also suspect that there are infinitely many Aaron numbers. In fact, suppose x is an integer such that

Table 1.

n	$S(n)$ $(= S(n+1))$	n	$S(n)$ $(= S(n+1))$
5	5	4191	141
8	6	5405	75
15	8	5560	150
77	18	5959	160
125	15	6867	122
714	29	8280	40
948	86	8463	54
1330	33	10647	39
1520	32	12351	205
1862	35	14587	532
2491	100	16932	107
3248	44	17080	79
4185	45	18490	93

$$s = 2x + 1$$
$$p = 8x + 5$$
$$q = 48x^2 + 24x - 1$$
$$r = 48x^2 + 30x - 1$$

are all primes. Then it is not hard to check that $pq + 1 = 4sr$ and $S(pq) = p + q = 4 + s + r = S(4sr)$, so that

$$pq = 384x^3 + 432x^2 + 112x - 5$$

is an Aaron number. So for example, when $x = 3$, we have $s = 7, p = 29, q = 503$, $r = 521$, and pq is the Aaron number 14587. We note that Schinzel's Conjecture H [1] implies that there are infinitely many integers x for which s, p, q, r are all primes. Hence this famous and widely believed conjecture would imply that there are infinitely many Aaron numbers. We checked the integers x in the interval $1 \leqslant x \leqslant 23900$ on the CDC 6400 and found that s, p, q, r were simultaneously prime for 18 values of x, the largest being $x = 23331$, giving rise to the Aaron number

$$4876994057472763.$$

The pair 714 and 715 has an additional peculiar property, namely

$$S(\sigma(714)) = S(\sigma(715))$$

where $\sigma(n)$ is the sum of the divisors of n. This property is also enjoyed by the Aaron pair 6867, 6868. However, 714 and 715 is the smallest such Aaron pair.

In closing, we wish to record several other unusual properties of 714 and 715. (Here φ is Euler's function.)

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	S(n) (= S(n + 1))
91	141
05	75
60	150
59	160
67	122
280	40
463	54
647	39
351	205
587	532
932	107
080	79
490	93

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$q + 1 = 4sr$ and $S(pq) = p + q =$

$12x - 5$

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property is also enjoyed by the
is the smallest such Aaron pair.
unusual properties of 714 and 715.

1. $\sigma(714)$ = a perfect cube
2. $\dfrac{\sigma(714)}{\varphi(714)}$ = a perfect square
3. $\varphi(\sigma(714)) = 2\varphi(\sigma(715))$ = a perfect square
4. 714 + 715 = 1429 = a backwards-forwards-sideways prime, in that 1429, 9241, 1249, 9421, 4129, and 4219 are all primes (not to mention that Columbus discovered America in 1492).

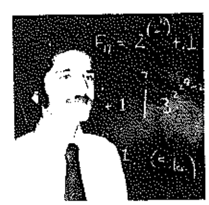
Reference

1. A. Schinzel and W. Sierpiński, "Sur certaines hypothèses concernant les nombres premiers," *Acta Arith.* 4 (1958), pp. 185-208.

ABOUT THE AUTHORS

Ms. Nelson, who was born in Memphis, Tennessee, in 1951, received her bachelor's degree from the University of Georgia in mathematics in 1973. She currently holds a graduate assistantship there, and is continuing her studies toward advanced degrees. She is particularly interested in applied mathematics, ordinary differential equations, and analysis, and regularly works summers teaching swimming in an Atlanta camp. She is fond of Russian authors, particularly Dostoevsky, and enjoys bicycling to and from campus in all sorts of weather.

Dr. Penney received his bachelor's in mathematics from Tulane in 1958, and his doctorate there in 1965—he wrote in knot theory under the direction of Professor Bruce Treybig. He worked part-time in biophysics research while at Tulane, and later as a teaching assistant at Tulane and as an instructor at L.S.U. in New Orleans. He joined the faculty of the Department of Mathematics of the University of Georgia in 1966, thereby returning to the state where he was born in 1938. His current interests include number theory, applied mathematics, and the use of the computer in mathematics. He is Associate Professor of Mathematics with publications in biophysics, topology, number theory, combinatorics, and applied mathematics.



Dr. Pomerance, who is currently Assistant Professor of Mathematics at the University of Georgia, was born in Missouri in 1944. He received his B.A. from Brown University in 1966 and his doctorate, under Professor John Tate, from Harvard University in 1972. His dissertation resulted in a new lower bound for the number of prime factors of an odd perfect number, and he is continuing his research in this and other areas of number theory.