Quaternions and conic sections: from algebra to geometry

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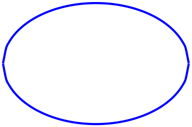
Wake Forest University
Mathematics Colloquium
Thursday 19 November 2009
4:15 - 5:15pm

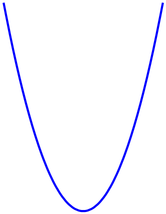
Conic sections

Apollonius, -3rd century

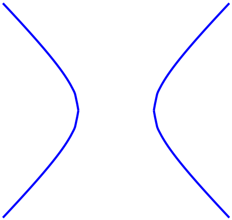
Ellipse



Parabola



Hyperbola



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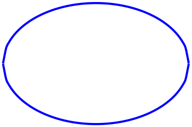

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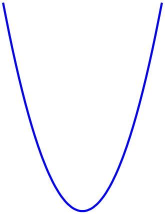
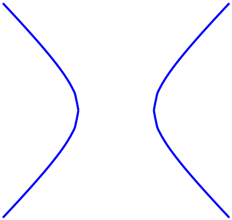


Hyperbola

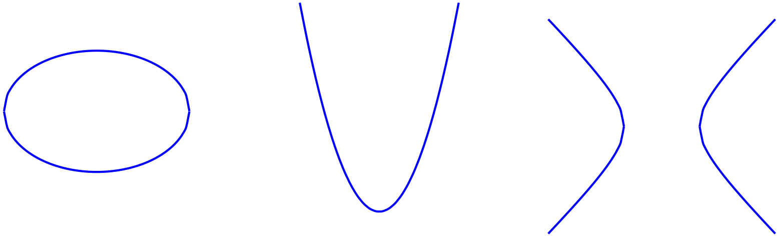


$$a\,x^2 + b\,xy + c\,y^2 + d\,x + e\,y + f = 0$$

Conic sections

Apollonius, -3rd century

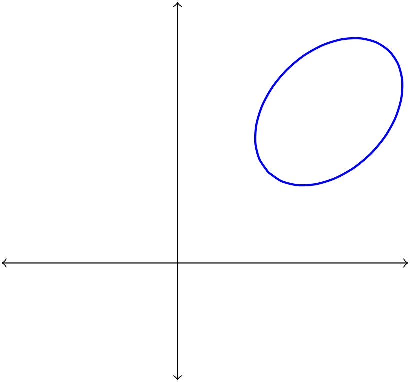
Ellipse | Parabola | Hyperbola



$b^2 - 4ac < 0$ | $b^2 - 4ac = 0$ | $b^2 - 4ac > 0$

$$a\,x^2 + b\,xy + c\,y^2 + d\,x + e\,y + f = 0$$

$$34x^2 - 32xy + 34y^2 + 128x - 272y + 319 = 0$$



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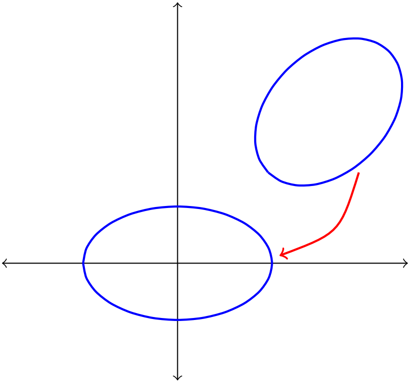

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

euclidean

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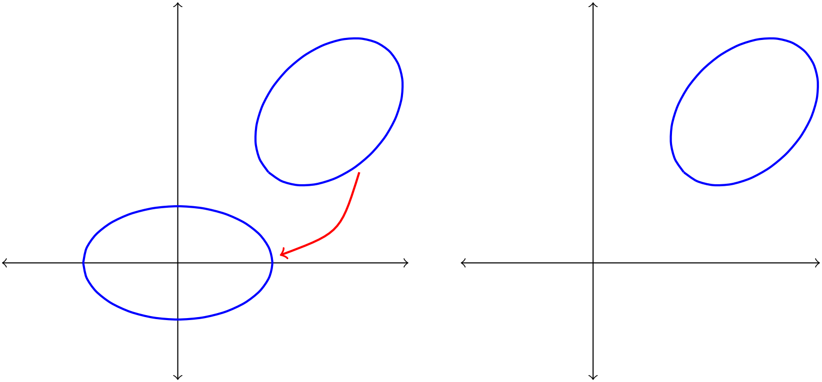

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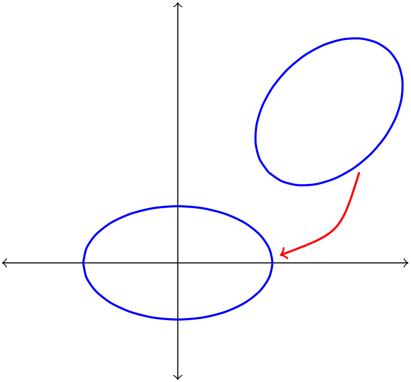

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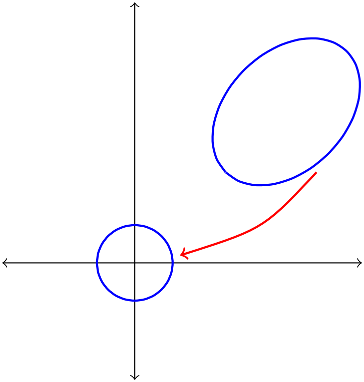
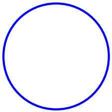


$$x^2 + y^2 = 1$$

general affine

General Affine Classification



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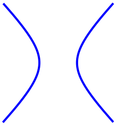
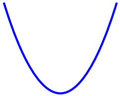
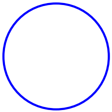
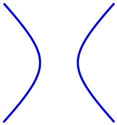
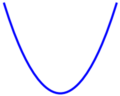


$x^2 - y^2 = 1$



$y = x^2$

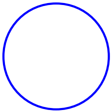
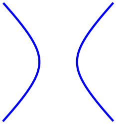
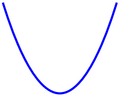
General Affine Classification

   $\varnothing$

$x^2 + y^2 = 1$ $\quad$ $x^2 - y^2 = 1$ $\quad$ $y = x^2$ $\quad$ $x^2 + y^2 = -1$

General Affine Classification

regular/degenerate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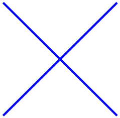

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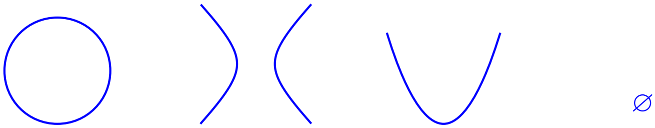

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Focus on regular conics.

General Affine Classification

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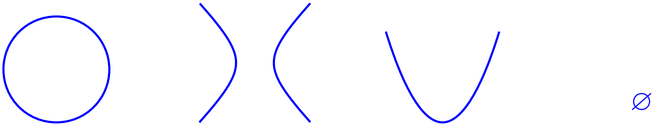

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Focus on regular conics. Current degenerate research!

Classification over $\mathbb{Q}$

$$\mathbb{Q} = \left\{ \frac{a}{b} \ : \ a,\ b \ \text{integers},\ b \neq 0 \right\}$$

$$q(x, y) = a\,x^2 + b\,xy + c\,y^2 + d\,x + e\,y + f \quad a, b, c, d, e, f \in \mathbb{Q}$$

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General affine equivalence of quadratic forms:

$$q(x,y) \approx q'(x,y) \ \Leftrightarrow \ q(x,y) = C \cdot q'(sx + ty + w, ux + vy + w')$$

$$= C \cdot q' \left(\begin{pmatrix} s & t \\ u & v \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w \\ w' \end{pmatrix} \right)$$

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Example: $x^2 + 2\,y^2 = 1 \ \overset{?}{\approx} \ x^2 + y^2 = 1$

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- $\mathrm{Sol}_{\mathbb{Q}}(x^2 + y^2 = 1) = \{(a,b,c) \in \mathbb{Z}^3 \;:\; a^2 + b^2 = c^2\}$
  (Pythagorean triples $(3,4,5)$, $(5,12,13)$, $\ldots$)

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Conclusion: $x^2 + 2\,y^2 = 1 \quad \not\approx \quad x^2 + y^2 = 1$ over $\mathbb{Q}$

Completing the Square

Theorem: Every regular conic section over $\mathbb{Q}$ is equivalent to:

$$y = x^2 \qquad \text{or} \qquad a\,x^2 + b\,y^2 = 1, \quad a, b \in \mathbb{Q}$$

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Properties: The Hasse-Witt symbol satisfies:

- $[a, b] \approx [b, a]$ $\qquad (x, y) \mapsto (y, x)$
- $[a, b] \approx [a, b\,c^2]$ $\qquad (x, y) \mapsto (x, cy)$
- $[a, -a] \approx [1, -1]$ $\qquad \binom{x}{y} \mapsto \frac{1}{4}\begin{pmatrix} a+1 & a-1 \\ a-1 & a+1 \end{pmatrix}\binom{x}{y}$

Quaternions

Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
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$$i^2 = j^2 = k^2 = ijk = -1$$

& cut it on a stone of this bridge.

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$$\mathbb{R} \quad \rightsquigarrow \quad \mathbb{C} = \mathbb{R} + i\,\mathbb{R} \quad \rightsquigarrow \quad \mathbb{H} = \mathbb{R} + i\,\mathbb{R} + j\,\mathbb{R} + k\,\mathbb{R}$$

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not ordered — not commutative

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$\mathbb{C}$: not ordered

$\mathbb{H}$: not commutative

$$ij = k, \quad jk = i, \quad ki = j, \quad ij = -ji, \quad ik = -ki, \quad jk = -kj$$

Skew-fields

Skew-field: $\mathbb{F}$ set with operations $+$ and $\cdot$ satisfying:

- Associativity: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
  $x + (y + z) = (x + y) + z$
- Distributivity: $x \cdot (y + z) = x \cdot y + x \cdot z$
- Identity: $0 + x = x = x + 0$
  $1 \cdot x = x = x \cdot 1$
- Inverses: $\exists -x, \quad x + (-x) = (-x) + x = 0$
  $x \neq 0 \quad \Rightarrow \quad \exists\, x^{-1}, \quad x \cdot x^{-1} = x^{-1} \cdot x = 1$
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- Commutativity: $x + y = y + x$
  $x \cdot y \neq y \cdot x$

Inverting Quaternions

$$(1 + i + j)\left(\frac{1}{3} - \frac{1}{3}i - \frac{1}{3}j\right) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + 0 = 1$$

Inverting Quaternions

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Quaternion conjugation:

$$(x+yi+zj+wk)(x-yi-zj-wk)=x^2+y^2+z^2+w^2$$

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Quaternion conjugation:

$$(x+yi+zj+wk)(x-yi-zj-wk)=x^2+y^2+z^2+w^2$$

$$\boxed{(x+yi+zj+wk)^{-1}=\frac{x-yi-zj-wk}{x^2+y^2+z^2+w^2}}$$

$$x^2+y^2+z^2+w^2=0 \quad \Leftrightarrow \quad x=y=z=w=0$$

Applications

Euclidean 3-space

Quantum Mechanics

Applications

Euclidean 3-space Imaginary quaternions

$$\mathbb{R}^3 \hookrightarrow \mathbb{H}$$

$$\vec{v} = (v_1, v_2, v_3) \mapsto v = v_1\, i + v_2\, j + v_3\, k$$

Quantum Mechanics

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$$v\, w = -\vec{v} \cdot \vec{w} + \vec{v} \times \vec{w}$$

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Quantum Mechanics Pauli matrices for fermionic spin (1920s):

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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$$i \leftrightarrow \sigma_1\sigma_2, \quad j \leftrightarrow \sigma_3\sigma_1, \quad k \leftrightarrow \sigma_2\sigma_3$$

$$S^3 = \{q \in \mathbb{H} \ : \ q\,\overline{q} = 1\} \to \mathbf{SO}(3)$$

$$q \mapsto v \mapsto q\, v\, q^{-1}$$

Quaternions over $\mathbb{Q}$

Theorem (Frobenius 1877): $\mathbb{F}$ a (skew-)field, $\mathbb{R} \subset \mathbb{F}$ center, then $\mathbb{F}$ is either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

Quaternions over $\mathbb{Q}$

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Theorem (Minkowski 1896, Merkurjev 1982): Every skew-field over $\mathbb{Q}$ is a Hilbert symbol $\mathbb{H}_{a,b}$.

Conics and Quaternions

Hasse-Wiit symbol $[a, b]$ and Hilbert symbol $\mathbb{H}_{a,b}$.

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Idea: Connection between 2-dimensional conic section

$$ax^2 + by^2 = 1$$

and 4-dimensional “quaternion invertibility” conic section

$$x^2 - ay^2 - bz^2 + abw^2 = 0$$