ADJOINT FUNCTORS IN ALGEBRAIC GEOMETRY

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Abstract. A collection of facts about adjoint various adjoint functors in algebraic geometry.

Let $f : Y \to X$ be a morphism of (topological) spaces. We have functors:

$$f_* : \mathsf{Sh}_{/Y} \to \mathsf{Sh}_{/X} \qquad \text{(pushforward)}$$
$$f^{-1} : \mathsf{Sh}_{/Y} \leftarrow \mathsf{Sh}_{/X} \qquad \text{(pullback)}$$

on categories of sheaves. Then in category theoretic terms:

f^{-1} is *left adjoint* to f_*
f_* is *right adjoint* to f^{-1}

Hom-set adjunction. There is an isomorphism of bifunctors:

$$\Phi : \mathrm{Hom}_{\mathsf{Sh}_{/Y}}(f^*-,-) \to \mathrm{Hom}_{\mathsf{Sh}_{/X}}(-,f_*-)$$

i.e. natural bijections

$$\Phi_{\mathscr{F},\mathscr{G}} : \mathrm{Hom}_{\mathsf{Sh}_{/Y}}(f^*\mathscr{F},\mathscr{G}) \to \mathrm{Hom}_{\mathsf{Sh}_{/X}}(\mathscr{F},f_*\mathscr{G})$$

for objects $\mathscr{F}$ in $\mathsf{Sh}_{/X}$ and $\mathscr{G}$ in $\mathsf{Sh}_{/Y}$.

(Co)unit adjunction. There are isomorphisms of functors:

$$\varepsilon : f^{-1} \circ f_* \to \mathrm{id}_{\mathsf{Sh}_{/Y}} \qquad \text{(counit)}$$
$$\eta : \mathrm{id}_{\mathsf{Sh}_{/X}} \to f_* \circ f^{-1} \qquad \text{(unit)}$$

satisfying

$$\mathrm{id}_{f^{-1}} = \varepsilon f^{-1} \circ f^{-1}\eta \quad : f^{-1} \xrightarrow{f^*\eta} f^{-1}f_*f^{-1} \xrightarrow{\varepsilon f^{-1}} f^{-1}$$
$$\mathrm{id}_{f_*} = f_*\varepsilon \circ \eta f_* \quad : f_* \xrightarrow{\eta f_*} f_*f^{-1}f_* \xrightarrow{f_*\varepsilon} f_*$$

i.e. natural morphisms

$$\varepsilon_{\mathscr{G}} : f^{-1}f_*\mathscr{G} \to \mathscr{G} \qquad \text{(counit)}$$
$$\eta_{\mathscr{F}} : \mathscr{F} \to f_*f^{-1}\mathscr{F} \qquad \text{(unit)}$$

satisfying

$$\mathrm{id}_{f^{-1}\mathscr{F}} = \varepsilon_{f^{-1}\mathscr{F}} \circ f^{-1}(\eta_{\mathscr{F}}) \quad : f^{-1}\mathscr{F} \xrightarrow{f^*(\eta_{\mathscr{F}})} f^{-1}f_*f^{-1}\mathscr{F} \xrightarrow{\varepsilon_{f^{-1}\mathscr{F}}} f^{-1}\mathscr{F}$$
$$\mathrm{id}_{f_*\mathscr{G}} = f_*(\varepsilon_{\mathscr{G}}) \circ \eta_{f_*\mathscr{G}} \quad : f_*\mathscr{G} \xrightarrow{\eta_{f_*}} f_*f^{-1}f_*\mathscr{G} \xrightarrow{f_*(\varepsilon_{\mathscr{G}})} f_*\mathscr{G}$$

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Date: November 15, 2010.