BRILL–NOETHER SPECIAL CUBIC FOURFOLDS OF DISCRIMINANT 14

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To Bill Fulton, on the occasion of his 80th birthday.

Abstract. We study the Brill–Noether theory of curves on K3 surfaces that are Hodge theoretically associated to cubic fourfolds of discriminant 14. We prove that any smooth curve in the polarization class has maximal Clifford index and deduce that a cubic fourfold contains disjoint planes if and only if it admits a Brill–Noether special associated K3 surface of degree 14. As an application, we prove that the complement of the pfaffian locus, inside the Noether–Lefschetz divisor $\mathcal{C}_{14}$ in the moduli space of cubic fourfolds, is contained in the irreducible locus of cubic fourfolds containing two disjoint planes.

Introduction

Let X be a *cubic fourfold*, i.e., a smooth cubic hypersurface $X \subset \mathbb{P}^5$ over the complex numbers. Determining the rationality of X is a classical question in algebraic geometry. Some classes of rational cubic fourfolds have been described by Fano [11] and Tregub [48], [49]. Beauville and Donagi [5] prove that *pfaffian* cubic fourfolds, i.e., those defined by pfaffians of skew-symmetric 6×6 matrices of linear forms, are rational. Hassett [19] describes, via lattice theory, Noether–Lefschetz divisors $\mathcal{C}_d$ in the moduli space $\mathcal{C}$ of smooth cubic fourfolds. A parameter count shows that $\mathcal{C}_{14}$ is the closure of the locus $\mathcal{P}f$ of pfaffian cubic fourfolds; Hodge theory shows (see [50, §3 Prop. 2]) that $\mathcal{C}_8$ is the locus of cubic fourfolds containing a plane. Hassett [18] identifies countably many divisors of $\mathcal{C}_8$ consisting of rational cubic fourfolds. Recently, Addington, Hassett, Tschinkel, and Várilly-Alvarado [2] identify countably many divisors of $\mathcal{C}_{18}$ consisting of rational cubic fourfolds, and Russo and Staglianò [45], [46] have shown that the very general cubic fourfolds in $\mathcal{C}_{26}$, $\mathcal{C}_{38}$, and $\mathcal{C}_{42}$ are rational. Nevertheless, it is expected that the very general cubic fourfold (as well as the very general cubic fourfold containing a plane) is not rational.

Short of a pfaffian presentation, how can one tell if a given cubic fourfold is pfaffian? Beauville [4] provides a homological criterion for a cubic hypersurface to be pfaffian, which for cubic fourfolds is equivalent to containing a quintic del Pezzo surface, but it is not clear how to translate this criterion into Hodge theory. More generally, how can one understand the complement $\mathcal{C}_{14} \smallsetminus \mathcal{P}f$ of the pfaffian locus? Such questions are implicit in [3] and [48], where cubic fourfolds with certain numerical properties are shown to be outside or inside, respectively, the pfaffian locus. In particular, Tregub studies the locus $\mathcal{C}_{\Pi}$ of cubic fourfolds that contain two disjoint planes, showing that this locus is irreducible of codimension 2 in $\mathcal{C}$, and that the general member does not contain a smooth quartic rational normal scroll nor a quintic del Pezzo surface, hence cannot be pfaffian. Our main result is that this is essentially all of the complement of the pfaffian locus.

Theorem 1. *The complement of the pfaffian locus $\mathcal{P}f$, inside the Noether–Lefschetz divisor $\mathcal{C}_{14}$ of the moduli space of cubic fourfolds, is contained in the irreducible locus $\mathcal{C}_{\Pi}$ of cubic fourfolds containing two disjoint planes.*

In other words, any $X \in \mathcal{C}_{14}$ is pfaffian or contains two disjoint planes (or both).

The proof combines several ingredients revolving around the Brill–Noether theory of special divisors on curves in K3 surfaces of degree 14. We use the determination, due to Mukai [40], [41] of the smooth projective curves C of genus 8 that are linear sections of the grassmannian $G(2,6) \subset \mathbb{P}^{14}$. This turns out to be equivalent to C lacking a g^2_7, equivalently, that C is Brill–Noether special. We also use a modified conjecture of Harris and Mumford, as proved by Green and Lazarsfeld [15], as well as the generalization due to Lelli-Chiesa [36], on line bundles on K3 surfaces computing the Clifford indices of smooth curves in a given linear system. We also need the earlier work of Saint-Donat [47] and Reid [43], [44], on hyperelliptic and trigonal linear systems on K3 surfaces, as well as useful refinements due to Knutsen [24], [25], [26] of the original result by Green and Lazarsfeld. Combining these results with lattice theory computations for cubic fourfolds and their associated K3 surfaces, as developed by Hassett [18], [19], we prove that the Clifford index of curves in the polarization class of any K3 surface of degree 14 associated to X must take the maximal value 3 (see Theorem 4.4), putting strong constraints on the geometry of cubic fourfolds in terms of the Brill–Noether theory of their associated K3 surfaces.

More generally, one might call a cubic fourfold X *Brill–Noether special* if X has an associated K3 surface S that is Brill–Noether special in the sense of Mukai [41, Def. 3.8], a condition implying that S has an ample divisor such that the general curve in its linear system is Brill–Noether special, see §1.2. Then our main result can be summarized by saying that a special cubic fourfold of discriminant 14 is Brill–Noether special if and only if it contains two disjoint planes. It would be interesting to study the Brill–Noether special loci in other divisors $\mathcal{C}_d$ of special cubic fourfolds, for example, in $\mathcal{C}_{26}$, $\mathcal{C}_{38}$, and $\mathcal{C}_{42}$. In the context of discriminant 26, Farkas and Verra [12] also appeal to the Brill–Noether theory of some associated K3 surfaces.

Our result, and more generally the ability to detect a pfaffian cubic fourfold via Hodge theory, has two immediate applications. First, we obtain a new explicit proof that every cubic fourfold in $\mathcal{C}_{14}$ is rational: Beauville and Donagi [5] prove that any pfaffian cubic fourfold is rational, and by a much more classical construction going back to Fano, every cubic fourfold containing disjoint planes is rational; this covers all cubic fourfolds in $\mathcal{C}_{14}$. This rationality result was initially obtained by Bolognesi, Russo, and Staglianò [6] using a much more classical approach involving one apparent double point surfaces, though this has been recently subsumed by the path-breaking work on the deformation invariance of rationality by Kontsevich and Tschinkel [28]. Second, we prove the existence of nonempty irreducible components of $\mathcal{P}f \cap \Pi$, which are necessarily of codimension ≥ 3 in $\mathcal{C}$. This immediately implies that the pfaffian locus is not Zariski open in $\mathcal{C}_{14}$. While this result was initially obtained in the course of conversations with M. Bolognesi and F. Russo based on the computer algebra calculations of G. Staglianò and earlier drafts of our respective papers, the proof presented in §5.1 does not require any explicit computer algebra computations (as opposed to the proof in [6]). However, it still seems plausible that the pfaffian locus is open inside the moduli space of marked cubic fourfolds of discriminant 14.

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1. Brill–Noether theory for polarized K3 surfaces of degree 14

All varieties are assumed to be over the complex numbers and all K3 surfaces are assumed to be smooth and projective.

1.1. **Grassmannians and curves of genus 8.** Let $G(2,6) \subset \mathbb{P}^{14}$ be the grassmannian of 2-planes in a 6-dimensional vector space, embedded in $\mathbb{P}^{14}$ via the Plücker embedding. It was classically known that a general flag of linear subspaces $P \subset Q$ of dimension 6 and 7 in $\mathbb{P}^{14}$ cut from $G(2,6)$ a K3 surface of degree 14 containing a canonical curve C of genus 8.

Recall that a g^r_d on a smooth projective curve C is a line bundle A of degree d with $h^0(C,A) \geq r+1$; it is *complete* if $h^0(C,A) = r+1$.

Theorem 1.1 (Mukai [40])**.** *A smooth projective curve C of genus 8 is a linear section of the Grassmannian $G(2,6) \subset \mathbb{P}^{14}$ if and only if C has no g^2_7.*

The Brill–Noether theorem states that when $\rho(g,r,d) = g-(r+1)(g-d+r)$ is negative, the general curve of genus g has no g^r_d. A curve supporting such a g^r_d is called *Brill–Noether special*. A curve not supporting any g^r_d whenever $\rho(g,r,d) < 0$ is called *Brill–Noether general*. When $\rho(g,r,d) = -1$, Eisenbud and Harris [10] proved that the locus of curves, in the moduli space $\mathcal{M}_g$ of curves of genus g, that support such a g^r_d, is irreducible of codimension 1. In particular, the locus of curves of genus 8 having a g^2_7 is of codimension 1 in $\mathcal{M}_8$.

The *Clifford index* of a line bundle A on a smooth projective curve C is the integer

$$\gamma(A) = \deg(A) - 2\,r(A),$$

where $r(A) = h^0(C,A) - 1$ is the *rank* of A. The Clifford index of C is

$$\gamma(C) = \min\{\, \gamma(A) \ : \ h^0(C,A) \geq 2 \text{ and } h^1(C,A) \geq 2 \,\}$$

and a line bundle A on C is said to compute the Clifford index of C if $\gamma(A) = \gamma(C)$. Clifford's theorem states that $\gamma(C) \geq 0$ with equality if and only if C is hyperelliptic; similarly $\gamma(C) = 1$ if and only if C is trigonal or a smooth plane quintic. At the other end, $\gamma(C) \leq \lfloor (g-1)/2 \rfloor$ with equality whenever C is Brill–Noether general.

Up to taking the adjoint line bundle $\omega_C \otimes A^\vee$, which has the same Clifford index, we can always assume that nontrivial special divisors g^r_d satisfy $1 \leq r \leq \lfloor (g-1)/2 \rfloor$

and $2 \leq d \leq g-1$. For $g = 8$, we list them for the convenience of the reader:

γ	3		2		1			0		
g^r_d	g^1_5	g^2_7	g^1_4	g^2_6	g^1_3	g^2_5	g^3_7	g^1_2	g^2_4	g^3_6
ρ	0	-1	-2	-4	-4	-7	-8	-6	-10	-12

In genus 8, the Brill–Noether special locus is controlled by the existence of a g^2_7.

Lemma 1.2. *A smooth projective curve C of genus 8 is Brill–Noether special if and only if it has a complete g^2_7.*

Proof. First note that if a curve has a complete g^r_d, then it has a complete g^k_d for all k between $d-g$ and r. Hence if C has a g^2_7 then it has a complete g^2_7. We can argue by the Clifford index. In Clifford index 3, the only special divisor is a g^2_7. For Clifford index 2, we use the facts that any genus 8 curve with a g^2_6 has a g^1_4 and any genus 8 curve with a g^1_4 has a g^2_7, see [40, Lemmas 3.4, 3.8]. In Clifford index 1, any genus 8 curve is trigonal, so taking twice the g^1_3 and adding a base point will result in a g^2_7. Finally, in Clifford index 0, the curve is hyperelliptic, so taking thrice the g^1_2 and adding a base point will result in a g^2_7. $\square$

1.2. **Brill–Noether theory for polarized K3 surfaces.** A *polarized K3 surface* (S, H) of degree d is a smooth projective K3 surface S together with a primitive ample line bundle H of self-intersection $d \geq 2$. If $C \subset S$ is a smooth irreducible curve in the linear system $|H|$, then $d = 2g-2$, where g is the genus of C. Following Mukai [41, Def. 3.8], we say that a polarized K3 surface (S, H) of degree $2g-2$ is *Brill–Noether general* if $h^0(S, H')\, h^0(S, H'') < h^0(S, H) = g+1$ for any nontrivial decomposition $H = H' \otimes H''$. Otherwise, we say *Brill–Noether special.* If a smooth irreducible curve $C \in |H|$ is Brill–Noether general then it follows that (S, H) is Brill–Noether general, cf. [23, Rem. 10.2]. While the converse is an open question in general, for low degrees it was checked by Mukai, using a case-by-case analysis.

Theorem 1.3. *A polarized K3 surface (S, H) of degree ≤ 18 or 22 is Brill–Noether general if and only if some smooth irreducible $C \in |H|$ is Brill–Noether general.*

Of course, we are mainly interested in the degree 14 case, where the results assembled below will suffice to prove the theorem.

The existence of special divisors on curves in a K3 surface was considered by Saint-Donat [47] and Reid [43], [44]. Harris and Mumford conjectured that the gonality of a curve should be constant in a linear system on a K3 surface. A counterexample was found by Donagi and Morrison [9] (in fact, this turned out to be the unique counterexample, cf. [7], [27]) and the conjecture was modified by Green [16, Conj. 5.8] to one about the constancy of the Clifford index in a linear system. In a similar spirit, one is interested in the question of when a given g^r_d on a curve in a K3 surface is the restriction of a line bundle from the K3. The conjecture of Green was proved in a celebrated paper by Green and Lazarsfeld.

Theorem 1.4 (Green–Lazarsfeld [15])**.** *Let S be a K3 surface and $C \subset S$ a smooth irreducible curve of genus $g \geq 2$. Then $\gamma(C') = \gamma(C)$ for every smooth curve $C' \in |C|$. Furthermore, if $\gamma(C) < \lfloor (g-1)/2 \rfloor$ then there exists a line bundle L on S whose restriction to any $C' \in |C|$ computes the Clifford index of C'.*

We can thus define the *Clifford index* $\gamma(S,H)$ of a polarized K3 surface (S,H) to be the Clifford index of any smooth irreducible curve $C \in |H|$, which is well-defined by Theorem 1.4.

In the case where (S,H) has degree 14, so that $C \in |H|$ has genus 8, we have that $\gamma(S,H) \leq 3$. If (S,H) is Brill–Noether general, then by Theorem 1.3, some curve $C \in |H|$ is Brill–Noether general (hence has maximal Clifford index), so that $\gamma(S,H) = 3$. When (S,H) is Brill–Noether special and $\gamma(S,H) < 3$, the result of Green and Lazarsfeld allows us to find a line bundle on S whose restriction to $C \in |H|$ computes the Clifford index. In fact, already for $\gamma(S,H) \leq 1$, results of Saint-Donat [47, Thm. 5.2] and Reid [43, Thm. 1] ensure that these line bundles can be chosen to be elliptic pencils, see §1.3 for details. Finally, when (S,H) is Brill–Noether special and $\gamma(S,H) = 3$, we would like to know if a g^2_7 on a curve $C \in |H|$ is the restriction of a line bundle on S. Since a g^2_7 has the generic Clifford index, we cannot appeal to the result of Green and Lazarsfeld. This situation, of Clifford general but not Brill–Noether general polarized K3 surfaces, is discussed more generally in [23, §10.2].

To this end, we have the following much more powerful result of Lelli–Chiesa, concerning when a specific g^r_d on a curve $C \subset S$ lying in a K3 surface is the restriction of a line bundle on S.

Theorem 1.5 (Lelli-Chiesa [36])**.** *Let S be a K3 surface and $C \subset S$ a smooth irreducible curve of genus $g \geq 2$ that is neither hyperelliptic nor trigonal. Let A be a complete g^r_d such that $r > 1$, $d \leq g-1$, $\rho(g,r,d) < 0$, and $\gamma(A) = \gamma(C)$. Assume that there is no irreducible genus 1 curve $E \subset S$ such that $E.C = 4$ and no irreducible genus 2 curve $B \subset S$ such that $B.C = 6$. Then A is the restriction of a globally generated line bundle L on S.*

This result comes from an in-depth study of generalized Lazarsfeld–Mukai bundles extending the original strategy of [15].

Remark 1.6. According to [36, Thm. 4.2*ff.*], the hypothesis on curves of genus 1 and 2 is completely satisfied as long as $\gamma(C) > 2$; otherwise, there is a list of seven exceptional cases when $\gamma(C) = 2$. We also remark that, according to the construction in the proof of [36, Thm. 4.2] (see also [35, Lemma 3.3] and [15, Lemma 3.1]), the line bundle L can be chosen to be globally generated, though this is not mentioned in the statement of the main theorem in [36].

When a K3 surface has Picard rank one, Lazarsfeld [34] has shown that the general curve in the linear system of the polarization class is Brill–Noether general. Hence Brill–Noether special K3 surfaces have higher Picard rank.

1.3. **Brill–Noether special K3 surfaces via lattice-polarizations.** Let Σ be an even nondegenerate lattice of signature $(1, \rho-1)$ with a distinguished class H of even norm $d > 0$. A Σ-*polarized* K3 surface is a polarized K3 surface (S,H) of degree d together with a primitive isometric embedding $\Sigma \hookrightarrow \mathrm{Pic}(S)$ preserving H. For a general discussion of lattice-polarized K3 surfaces and their moduli, see [8]. In particular, there exists a quasi-projective coarse moduli space $\mathcal{K}_\Sigma$ of dimension $20-\rho$ and a forgetful morphism $\mathcal{K}_\Sigma \to \mathcal{K}_d$ to the moduli space of polarized K3 surfaces of degree d. The main result of this section is the following characterization of Brill–Noether special K3 surfaces of degree 14 via lattice polarizations. The same result is obtained by Greer, Li, and Tian [17] using a different calculation.

$\gamma = 0$	$\gamma = 1$	$\gamma = 2$		$\gamma = 3$
$\begin{array}{c\|cc} & H & E \\ \hline H & 14 & 2 \\ E & 2 & 0 \end{array}$	$\begin{array}{c\|cc} & H & E \\ \hline H & 14 & 3 \\ E & 3 & 0 \end{array}$	$\begin{array}{c\|cc} & H & E \\ \hline H & 14 & 4 \\ E & 4 & 0 \end{array}$	$\begin{array}{c\|cc} & H & L \\ \hline H & 14 & 6 \\ L & 6 & 2 \end{array}$	$\begin{array}{c\|cc} & H & L \\ \hline H & 14 & 7 \\ L & 7 & 2 \end{array}$
$d_S = -4$ $d_S^0 = -14$	$d_S = -9$ $d_S^0 = -14 \cdot 9$	$d_S = -16$ $d_S^0 = -14 \cdot 4$	$d_S = -8$ $d_S^0 = -14 \cdot 2$	$d_S = -21$ $d_S^0 = -6$
$(b, c) = (6, 8)$	$(b, c) = (5, 6)$	$(b, c) = (2, 2)$	$(b, c) = (4, 4)$	$(b, c) = (7, 12)$

Table 1. Lattices embedded in Brill–Noether special K3 surfaces of degree 14 and Clifford index γ. Here, d and d_0 denote the discriminants of the lattice and of $\langle H \rangle^\perp$, respectively. The pair (b, c) refers to the unique rank 3 cubic fourfold lattice, whose associated lattice is the given one, normalized as in Proposition 3.1.

Theorem 1.7. *If a polarized K3 surface (S, H) of degree 14 is Brill–Noether special then it admits a lattice polarization for one of the five rank 2 lattice appearing in Table 1 and $\gamma(S, H)$ is bounded above by the corresponding value of γ on the table. In particular, the Brill–Noether special locus in $\mathcal{K}_{14}$ is the union of five divisors indexed by the lattices in Table 1.*

Before the proof of the theorem, we need some lemmas on elliptic pencils on K3 surfaces, which are mostly contained in the work of Saint-Donat [47] and Knutsen [25], [26]. By an *elliptic pencil* we mean a line bundle E on a K3 surface S such that the generic member of the linear system $|E|$ is a smooth genus one curve. A result of Saint-Donat [47, Prop. 2.6(ii)] says that if E is generated by global sections and $E^2 = 0$, then E is a multiple of an elliptic pencil. If E is an elliptic pencil then E is primitive in $\mathrm{Pic}(S)$ (cf. [22, Ch. 2, Remark 3.13(i)]), $E^2 = 0$, $h^0(S, E) = 2$, and $h^1(S, E) = 0$.

Lemma 1.8. *Let (S, H) be a polarized K3 surface of degree $2g - 2 \geq 2$, let $C \in |H|$ be a smooth irreducible curve, and let E be a globally generated line bundle on S with $E^2 = 0$ and $E.C = d < 2g - 2$. Then $E|_C$ is a g^1_d if and only if E is an elliptic pencil such that $h^1(S, E(-C)) = 0$.*

Proof. First remark that since H is base point free and $(E - C).C = d - (2g - 2) < 0$ by hypothesis, we get that $h^0(S, E(-C)) = 0$, cf. [26, Proof of Prop. 2.1].

Now, assume that E is an elliptic pencil and that $h^1(S, E(-C)) = 0$. Then the long exact sequence in cohomology associated to the exact sequence of sheaves

$$0 \to E(-C) \to E \to E|_C \to 0$$

together with the fact that $h^0(S, E) = 2$, implies that $h^0(C, E|_C) = 2$. Since $\deg(E|_C) = E.C = d$, we have that $E|_C$ is a g^1_d.

Now assume that $E|_C$ is a g^1_d. By Saint-Donat [47, Prop. 2.6(ii)], $E = F^{\otimes k}$ for an elliptic pencil F and some $k \geq 1$ dividing d. Again considering the same long exact sequence as above, the last terms, when rewritten using Serre duality and the fact that $H^0(S, E^\vee) = 0$ since E is effective, read

$$H^1(S, E|_C) \to H^0(S, E(-C)^\vee) \to 0.$$

By Riemann–Roch on C, we have $h^1(S, E|_C) = h^1(C, E|_C) = 2 - (d - g + 1)$, hence $h^0(S, E(-C)^\vee) \leq 2 - (d - g + 1)$. By Riemann–Roch on S, we have

$$h^0(S, E(-C)^\vee) - h^1(E(-C)) = 2 + \frac{1}{2}(E - C)^2 = 2 + \frac{1}{2}(-2d + 2g - 2) = 2 - (d - g + 1),$$

using Serre duality and the fact that $H^0(S, E(-C)) = 0$, hence $h^0(S, E(-C)^\vee) \geq 2 - (d - g + 1)$ and $h^1(S, E(-C)) = 0$. However, the beginning terms of the long exact sequence read

$$0 \to H^0(S, E) \to H^0(S, E|_C) \to H^1(E(-C))$$

implying that $h^0(S, E) = h^0(C, E|_C) = 2$ (since $E|_C$ is a g^1_e). But $h^0(S, E) = k + 1$ and thus we conclude that $k = 1$, i.e., E is an elliptic pencil. $\square$

Proof of Theorem 1.7. Let $C \subset S$ be a smooth irreducible curve (of genus 8) in the linear system of H. We argue by the Clifford index of (S, H), equivalently, of C.

If $\gamma(C) = 0$, i.e., C is hyperelliptic by Clifford's Theorem, then by Saint-Donat [47, Thm. 5.2] (cf. Reid [43, Prop. 3.1]), the g^1_2 on C is the restriction of an elliptic pencil E such that $E.H = 2$.

If $\gamma(C) = 1$, i.e., C is trigonal, then by Reid [44, Thm. 1] (cf. [47, Thm. 7.2]), after verifying $8 > \frac{1}{4}3^2 + 3 + 2$, the g^1_3 on C is the restriction of an elliptic pencil E such that $E.H = 3$.

In these first two cases, the sublattice of $\mathrm{Pic}(S)$ generated by H and E is primitive. Indeed, if not, then this sublattice admits a finite index overlattice contained in $\mathrm{Pic}(S)$. However, using the correspondence between finite index overlattices and isotropic subgroups of the discriminant form (cf., Nikulin [42, §1.4]), we find that, in this case, the only finite index overlattice would admit a class $F \in \mathrm{Pic}(S)$, where $eF = E$, for $e = 2$ or 3, respectively. However, as E is an elliptic pencil on S, it is a primitive class in $\mathrm{Pic}(S)$, hence no such overlattice exists.

If $\gamma(C) = 2$, then by Green–Lazarsfeld [15] (since the generic value of the Clifford index is 3, see Theorem 1.4), there is a line bundle L on S such that $L|_C$ is a g^1_4 or a g^2_6. Then $L.H = \deg(L|_C) = 4$ or 6, respectively. Furthermore, by a result of Knutsen [24, Lemma 8.3], we can choose L satisfying

$$0 \leq L^2 \leq 4 \quad \text{and} \quad 2L^2 \leq L.H \quad \text{and} \quad 2 = L.H - L^2 - 2$$

with $L^2 = 4$ or $2L^2 = L.H$ if and only if $H = 2L$. However, since 14 is squarefree, $H = 2L$ is impossible, hence the only possibilities are that $L^2 = 0$ and $L.H = 4$, $L^2 = 0$ and $L.H = 6$, or $L^2 = 2$ and $L.H = 6$. As a consequence of Martens' proof [38] of the main result of [15] (cf. proof of [24, Lemma 8.3]), we can also choose L generated by global sections and with $h^1(S, L(-C)) = 0$. Suggestively, in the two former cases, we denote L by E.

We now argue that the case $E^2 = 0$ and $E.H = 6$ is impossible. First assume that E is an elliptic pencil. Lemma 1.8 then implies that $E|_C$ is a g^1_6, contradicting the assumption that it is a g^2_6. Hence E cannot be an elliptic pencil. Thus by the result of Saint-Donat mentioned above, $E = kF$ for $k = 2, 3, 6$ and an elliptic pencil F. The case $k = 6$ is impossible, since $F^2 = 0$ and $F.H = 1$ contradicts the ampleness of H. For $k = 2, 3$, we have $F^2 = 0$ and $F.H = 6/k \leq 3$, so that results of Saint-Donat [47, Prop. 5.2, 7.15] imply that $F|_C$ is a $g^1_{6/k}$, contradicting the fact that $\gamma(C) = 2$.

In the remaining two cases, we argue that the sublattice of $\mathrm{Pic}(S)$ generated by H and E (resp. H and L) is primitive. As before, we appeal to the correspondence between finite index overlattices and isotropic subgroups of the discriminant form (cf., Nikulin [42, §1.4]). In the case $E^2 = 0$ and $E.H = 4$, the only finite overlattice would contain a class dividing E, however since $E|_C$ must be a g^1_4, then by Lemma 1.8, E is an elliptic pencil and is thus a primitive class in $\mathrm{Pic}(S)$. Hence, the sublattice generated by H and E is primitive. In the case $L^2 = 2$ and $L.H = 6$, the only finite index overlattice would contain a class $F \in \mathrm{Pic}(S)$ such that $2F = H - L$, however, such F would then satisfy $F^2 = (H-L)^2/4 = 1$, which is impossible since $\mathrm{Pic}(S)$ is an even lattice. Hence, the sublattice generated by H and L is primitive.

Finally, assume that $\gamma(C) = 3$. Then all the hypotheses of the results of Lelli-Chiesa [36] (see Theorem 1.5) are satisfied, hence there exists a line bundle L on S such that $L|_C$ is a g^2_7. In particular, $L.C = \deg(L|_C) = 7$. As before, by Remark 1.6, L can be chosen to be globally generated, so that $2n = L^2 \geq 0$. Furthermore, by [23, Prop. 10.5], we can choose L so that $L^2 = 2$. The sublattice of $\mathrm{Pic}(S)$ generated by H and L is then primitive since its discriminant is squarefree.

Thus in each case, the polarized K3 surface (S, H) has a lattice-polarization with respect to one of the lattices on Table 1. □

Remark 1.9. Every smooth curve C of genus 8 contains a finite number of g^1_5 divisors. If $\gamma(C) = 3$ and C lies on a K3 surface S with a primitive degree 14 polarization H, then it could happen that none of the g^1_5 divisors are the restriction of a line bundle from S (e.g., the corresponding Lazarsfeld–Mukai bundles are simple). However, if a g^1_5 is the restriction of a line bundle on S, then arguing as in the proof of Theorem 1.7, one can verify that the Picard lattice of S admits a primitive sublattice generated by H and E, where E is an elliptic pencil such that $H.E = 5$ and $E|_C$ is the g^1_5.

2. Lattice polarized cubic fourfolds

Let X be a smooth cubic fourfold and let $A(X)$ denote the lattice of codimension 2 algebraic cycles $\mathrm{CH}^2(X)$ with its usual intersection form. Then via the cycle class map, $A(X)$ is isomorphic to $H^4(X,\mathbb{Z}) \cap H^{2,2}(X)$ by the validity of the integral Hodge conjecture for cubic fourfolds proved by Voisin [51].

Given a positive definite lattice Λ containing a distinguished element h^2 of norm 3, a Λ-*polarized* cubic fourfold is a cubic fourfold X together with the data of a primitive isometric embedding $\Lambda \hookrightarrow A(X)$ preserving h^2. The main results of Looijenga [37] and Laza [33] on the description of the period map for cubic fourfolds imply that smooth Λ-polarized cubic fourfolds exist if and only if Λ admits a primitive embedding into $H^2(X,\mathbb{Z}) = \langle 1 \rangle^{\oplus 21} \oplus \langle -1 \rangle^{\oplus 2}$ and Λ contains no short roots (i.e., elements $v \in \Lambda$ with norm 2) nor long roots (i.e., elements $v \in \Lambda$ with norm 6 such that $v.h^2 = 0$ and $v.\langle h^2 \rangle^{\perp} \subset 3\mathbb{Z}$). We call any such lattice Λ a *cubic fourfold lattice*.

For a cubic fourfold lattice Λ of rank ρ, an adaptation of the argument of Hassett [19, Thm. 3.1.2] (see also [20, §2.3]) proves that the moduli space $\mathcal{C}_\Lambda$ of Λ-polarized cubic fourfolds is a quasi-projective variety of dimension $21 - \rho$. There is a forgetful map $\mathcal{C}_\Lambda \to \mathcal{C}$, whose image we denote by $\mathcal{C}_{[\Lambda]}$. In other words, $\mathcal{C}_{[\Lambda]} \subset \mathcal{C}$ is the locus of cubic fourfolds X such that $A(X)$ admits a primitive isometric embedding of Λ preserving h^2. We remark that the forgetful map $\mathcal{C}_\Lambda \to \mathcal{C}_{[\Lambda]}$ is generically finite to one, and whose degree depends on the number of automorphisms of Λ fixing h^2.

The possible rank 2 cubic fourfold lattices were classified by Hassett [19]; such a lattice K_d is uniquely determined by its discriminant, which can be any number $d > 6$ such that $d \equiv 0, 2 \pmod 6$. Then $\mathcal{C}_{K_d}$ coincides with the moduli space $\mathcal{C}_d^{\mathrm{mar}}$ of *marked* special cubic fourfolds of discriminant d considered by Hassett [19, §5.2] and $\mathcal{C}_{[K_d]}$ coincides with the Noether–Lefschetz divisor $\mathcal{C}_d \subset \mathcal{C}$. For cubic fourfold lattices Λ of rank 3, the loci $\mathcal{C}_{[\Lambda]}$ were considered in [1], [3], [6], [14], [48], [49].

Given a primitive embedding $\Lambda \hookrightarrow \Lambda'$ of cubic fourfold lattices preserving h^2, there is an induced morphism $\mathcal{C}_{\Lambda'} \to \mathcal{C}_\Lambda$ and an inclusion of subvarieties $\mathcal{C}_{[\Lambda']} \subset \mathcal{C}_{[\Lambda]}$. In particular, we have that $\mathcal{C}_{[\Lambda]} \subset \mathcal{C}_d$ whenever Λ admits a primitive embedding of K_d preserving h^2.

When $\Lambda = \Pi$ is the lattice with Gram matrix

$$
\begin{array}{c|ccc}
 & h^2 & T & P \\
\hline
h^2 & 3 & 4 & 1 \\
T & 4 & 10 & -1 \\
P & 1 & -1 & 3
\end{array}
\quad \cong \quad
\begin{array}{c|ccc}
 & h^2 & P & P' \\
\hline
h^2 & 3 & 1 & 1 \\
P & 1 & 3 & 0 \\
P' & 1 & 0 & 3
\end{array}
\tag{1}
$$

with the isomorphism defined by $T = 2h^2 - P - P'$, then $\mathcal{C}_{[\Pi]}$ is one of the most well-studied codimension 2 loci in the moduli space of cubic fourfolds, cf. [11], [48], [50, §3, App.].

Proposition 2.1. *The subvariety $\mathcal{C}_{[\Pi]} \subset \mathcal{C}$ is an irreducible component of $\mathcal{C}_8 \cap \mathcal{C}_{14}$ and coincides with the locus of cubic fourfolds that contain disjoint planes.*

Proof. The proof of the first statement is in [3, Thm. 4], cf. [11], [48]. The existence of two disjoint planes follows from the proof given in Voisin [50, §3, App., Prop.] and the refinement due to Hassett [19, §3]. □

Fix an *admissible discriminant* $d > 6$, i.e., such that $d \equiv 0, 2 \bmod 6$ and such that $4 \nmid d$, $9 \nmid d$, and $p \nmid d$ for any odd prime $p \equiv 2 \bmod 3$. Hassett [19, §5] proves that for any cubic fourfold X with a marking of discriminant d, the orthogonal complement $K_d^\perp$ of K_d inside $H^4(X, \mathbb{Z})$ is Hodge isometric to a twist $\mathrm{Pic}(S)_0(-1)$ of the primitive cohomology lattice of a polarized K3 surface (S, H) of degree d, and that such a Hodge-theoretic association gives rise to a choice of open immersion $\mathcal{C}_{K_d} = \mathcal{C}_d^{\mathrm{mar}} \hookrightarrow \mathcal{K}_d$ of moduli spaces (cf. [19, Corollary 5.2.4]). The choice of such an open immersion is determined by an isomorphism between the discriminant forms of the abstract lattices $K_d^\perp$ and $\mathrm{Pic}(S)_0(-1)$, modulo scaling by $\{\pm 1\}$; there are 2^{r-1} such choices, where r is the number of distinct odd primes dividing d, see [19, Corollary 5.2.4], [20, Proposition 26].

Now, given a cubic fourfold lattice Λ and a fixed primitive embedding $K_d \hookrightarrow \Lambda$ preserving h^2, we are interested in generalizing this open immersion to Λ-polarized cubic fourfolds. We can do this explicitly in the case of interest to us, namely when $d = 14$ and the rank of Λ is 3, due to the following lemma.

Lemma 2.2. *Let Λ be a rank 3 cubic fourfold lattice with a fixed primitive embedding $K_{14} \hookrightarrow \Lambda$ preserving h^2. Then, up to isometry, there is a unique rank 2 even indefinite lattice $\sigma(\Lambda)$ with discriminant $-d(\Lambda)$, a distinguished class H of norm d, and such that the orthogonal complement of K_{14} in Λ is isometric (up to twist) with the orthogonal complement of H in $\sigma(\Lambda)$.*

Proof. As the sublattice $K_{14} = \langle h^2, T\rangle \subset \Lambda$ is primitive, there exists a class $J \in \Lambda$ and integers a, b, c such that

$$\begin{array}{c|ccc} & h^2 & T & J \\ \hline h^2 & 3 & 4 & a \\ T & 4 & 10 & b \\ J & a & b & c \end{array}$$

By translating J to $J - a(T - h^2)$, we can assume that $a = 0$. Directly computing the determinant of this Gram matrix, we then find that $d(\Lambda) = -3b^2 + 14c \equiv (5b)^2$ is a square modulo 14. Let $0 \leq \alpha \leq 7$ be such that $\alpha^2 \equiv d(\Lambda)$ modulo 14. Then we can write $d(\Lambda) = \alpha^2 - 14\beta$ for some integer β. Now we argue that β is even. Since J is orthogonal to h^2 and $\langle h^2\rangle^\perp$ is an even lattice, we have that $J^2 = c$ must be even. Thus $d(\Lambda) \equiv 0, 1 \pmod 4$. From the equation $d(\Lambda) = \alpha^2 - 14\beta$ we see that $d(\Lambda)$ and α have the same parity, and by looking modulo 4, we finally find that β must be even.

We now define $\sigma(\Lambda)$ to be the rank 2 lattice $\langle H, L\rangle$ with Gram matrix

$$\begin{array}{c|cc} & H & L \\ \hline H & 14 & \alpha \\ L & \alpha & \beta \end{array}$$

Then $d(\sigma(\Lambda)) = 14\beta - \alpha^2 = -d(\Lambda)$ and hence $\sigma(\Lambda)$ is an indefinite even lattice since $d(\Lambda) > 0$ and β is even.

We now directly calculate that the orthogonal complement of K_{14} in Λ is generated by $(4bh^2 - 3bT + 14J)/\gcd(b, 14)$ and that the orthogonal complement of H in $\sigma(\Lambda)$ is generated by $(\alpha H - 14L)/\gcd(\alpha, 14)$. Computing the self-intersections of these generators yields $14d(\Lambda)/\gcd(b, 14)^2$ and $-14d(\sigma(\Lambda))/\gcd(\alpha, 14)^2$, respectively. Noting that $\alpha \equiv \pm 5b$ modulo 14, we have that $\gcd(b, 14) = \gcd(\alpha, 14)$, which proves the claim about the isometry of orthogonal complements.

Finally, we remark that $\sigma(\Lambda)$ is unique up to isometry with these properties. Indeed, given any rank 2 even indefinite lattice with Gram matrix as above, after a translation and a possible reflection, we can always choose $0 \leq \alpha \leq 7$. But then α, β are uniquely determined by the equation $14\beta - \alpha^2 = -d(\Lambda)$. So $\sigma(\Lambda)$ is unique up to isometry. $\square$

The proof of Lemma 2.2 provides an algorithm, given the Gram matrix of Λ, to calculate a Gram matrix of $\sigma(\Lambda)$. As an example, we calculate that the Gram matrix of $\sigma(\Pi)$ is

$$\begin{array}{c|cc} & H & E \\ \hline H & 14 & 7 \\ E & 7 & 2 \end{array} \tag{2}$$

where Π is the lattice in (1), with fixed primitive embedding $K_{14} = \langle h^2, T\rangle \hookrightarrow \Pi$.

Now, for any rank 3 cubic fourfold lattice Λ with a fixed choice of primitive embedding $K_{14} \hookrightarrow \Lambda$ as in Lemma 2.2, consider the moduli space $\mathcal{K}_{\sigma(\Lambda)}$ of $\sigma(\Lambda)$-polarized K3 surfaces and the forgetful morphism $\mathcal{K}_{\sigma(\Lambda)} \to \mathcal{K}_{14}$, whose image is a divisor $\mathcal{K}_{[\sigma(\Lambda)]} \subset \mathcal{K}_{14}$. For any Λ-polarized cubic fourfold X, the fixed primitive embedding $K_{14} \hookrightarrow \Lambda$ determines a discriminant 14 marking of X, which induces an associated polarized K3 surface (S, H) of discriminant 14 admitting a $\sigma(\Lambda)$-polarization by Lemma 2.2. We recall that for discriminant 14, there is a unique choice of open immersion $\mathcal{C}_{K_{14}} \hookrightarrow \mathcal{K}_{14}$, see [19, §6]. Then following Hassett [19, §5.2], we have the following.

Proposition 2.3. *Let Λ be a rank 3 cubic fourfold lattice with a fixed primitive embedding $K_{14} \hookrightarrow \Lambda$ preserving h^2. Then there exists an open immersion $\mathcal{C}_\Lambda \hookrightarrow \mathcal{K}_{\sigma(\Lambda)}$ of moduli spaces and a commutative diagram*

$$\begin{tikzcd} \mathcal{C}_{K_{14}} \arrow[r, hook] & \mathcal{K}_{14} \\ \mathcal{C}_\Lambda \arrow[u] \arrow[r, hook] & \mathcal{K}_{\sigma(\Lambda)} \arrow[u] \end{tikzcd}$$

where the vertical arrows are the forgetful maps and the top horizontal arrow is the (unique choice of) open immersion constructed by Hassett.

3. Cubic fourfold lattice normal forms

This section is devoted to establishing normal forms for cubic fourfold lattices of rank 3 with a discriminant 14 marking and their associated K3 surface Picard lattices.

Proposition 3.1. *Let Λ be a rank 3 cubic fourfold lattice with a fixed primitive embedding of K_{14} preserving h^2. Then there exists a basis h^2, T, J of Λ with respect to which Λ has Gram matrix*

$$\begin{array}{c|ccc} & h^2 & T & J \\ \hline h^2 & 3 & 4 & 0 \\ T & 4 & 10 & b \\ J & 0 & b & c \end{array} \tag{3}$$

for some integers $0 \le b \le 7$ and $c > \max(2, 3b^2/14)$ even.

Proof. Just as in the proof of Lemma 2.2, we can choose the primitive sublattice $K_{14} = \langle h^2, T\rangle$, and then there exists a class $J \in \Lambda$ and integers b, c such that that Gram matrix of Λ has the shape (3). Since $(3T - 4h^2).h^2 = 0$, we can further translate J to $\pm J - m(3T - 4h^2)$, which preserves $h^2.J = 0$ and allows us modify b modulo $14 = (3T - 4h^2).T$ and up to sign, so we can choose representatives $0 \le b \le 7$. Being a primitive sublattice of $A(X)$, we know that Λ is positive definite, hence its discriminant $-3b^2 + 14c$ must be positive, which forces $c > 3b^2/14$. Already in the proof of Lemma 2.2, we saw that c must be even; also c must be greater than 2, since $\langle h^2\rangle^\perp$ is an even lattice with no roots (i.e., vectors with norm 2). Note that a similar normal form analysis is carried out in [3, §2], [1, Lemma 4.2]. □

One application of the normal form in Proposition 3.1 is that the lattice $\sigma(\Lambda)$ can be even more explicitly computed from Λ. Given (b, c) that determine Λ, we compute that $\sigma(\Lambda)$ has Gram matrix

$$\begin{array}{c|cc} & H & L \\ \hline H & 14 & 2b \\ L & 2b & \frac{b^2}{2} - c \end{array} \quad \text{or} \quad \begin{array}{c|cc} & H & L \\ \hline H & 14 & 7-2b \\ L & 7-2b & \frac{b^2-4b+7}{2} - c \end{array}$$

depending on whether b is even or odd, respectively.

A consequence of this calculation is that $\sigma(\Lambda)$ together with H determines the pair (b, c), and hence Λ together with the fixed primitively embedded K_{14} up to isomorphism. In the last line of Table 1, we have recorded, by listing the pair (b, c), the unique rank 3 cubic fourfold lattices Λ with a primitive embedding $K_{14} \hookrightarrow \Lambda$ whose associated lattice is the given $\sigma(\Lambda)$.

Finally, we are in a position to deduce the following.

Proposition 3.2. *If a smooth cubic fourfold X has an associated K3 surface (S, H) of degree 14 that is Brill–Noether special and with Clifford index 3, then $X \in \mathcal{C}_{[\Pi]}$.*

Proof. The cubic fourfold X, together with the discriminant 14 marking $K_{14} \hookrightarrow A(X)$ whose associated K3 surface is (S, H), determines a point on the moduli space $\mathcal{C}_{K_{14}}$. Under the open immersion $\mathcal{C}_{K_{14}} \hookrightarrow \mathcal{K}_{14}$ constructed by Hassett, this point maps to (S, H). Since (S, H) is Brill–Noether special with Clifford index 3, Theorem 1.7 implies that S admits a Σ-polarization where Σ is the lattice in the $\gamma = 3$ column of Table 1. Hence (S, H) determines a point on the moduli space $\mathcal{K}_\Sigma$. Via the Hodge isometry $K_d^\perp \cong \mathrm{Pic}(S)_0(-1)$, we lift $\Sigma \cap \mathrm{Pic}(S)_0(-1)$ to a primitive rank 3 lattice $\Lambda \subset A(X)$; this is nothing but the saturation of the sum of K_d and the image of $\Sigma \cap \mathrm{Pic}(S)_0(-1)$ in $A(X)$. A calculation of this saturation shows that, in fact, $\Lambda \cong \Pi$. Recalling that $\Sigma = \sigma(\Pi)$ by (2), we thus have that the moduli point of (S, H) in $\mathcal{K}_{\sigma(\Pi)}$ is in the image of the open immersion $\mathcal{C}_\Pi \hookrightarrow \mathcal{K}_{\sigma(\Pi)}$, which finishes the proof by the commutativity of the diagram in Proposition 2.3. $\square$

In fact, later on in Theorem 4.4, we will show that for any cubic fourfold X with a discriminant 14 marking, the polarized K3 surface (S, H) of degree 14 Hodge theoretically associated to X always has Clifford index 3.

We end this section with some lattice computations that will be useful later on.

Let $\Lambda = (\mathbb{Z}^n, b)$ be an integral nondegenerate lattice with a choice of standard basis and bilinear form $b : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ with Gram matrix B and $d(\Lambda) = \det(B)$. Then with respect to the dual standard basis, the dual lattice $\Lambda^\vee = (\mathbb{Z}^n, b^\vee)$ can be considered as a bilinear form $b^\vee : \mathbb{Z}^n \times \mathbb{Z}^n \to \frac{1}{d}\mathbb{Z}$ having Gram matrix B^{-1}. The canonical isometric embedding $\Lambda \to \Lambda^\vee$ is then identified with the matrix multiplication map $B : (\mathbb{Z}^n, b) \to (\mathbb{Z}^n, b^\vee)$. In particular, for $\alpha \in \Lambda^\vee$ and $a \in \Lambda$, we have that $\alpha(a) = \alpha \cdot a$ is the usual dot product with respect to these identifications.

As a consequence, if h^2 is the first element of a basis of $A(X)$, then $v \in A(X)$ is a *long root* if and only if $h^2.v = 0$, $v.v = 6$, and all but the first coordinate of v is divisible by 3.

With this, we can see that if $\Lambda \hookrightarrow \Lambda'$ is a primitive embedding of lattices preserving a distinguished element h^2, and Λ has a long root, then Λ' also has a long root (since we can extend any long root of Λ by 0 to a long root of Λ'). Clearly, any short root of Λ is a short root of Λ'.

We state some useful, if not easy, necessary conditions for a lattice to occur as the intersection lattice of a smooth cubic fourfold.

Lemma 3.3. *No lattice of the following type can arise as the intersection lattice $A(X)$ of a smooth cubic fourfold X:*

(1) *A unimodular lattice.*
(2) *A lattice with odd rank $\rho \leq 11$ and discriminant a prime $p \equiv \rho \bmod 4$.*
(3) *A lattice with odd rank $\rho \leq 11$ and discriminant exactly divisible by 2.*

Proof. For (1), it is a consequence of the classification of unimodular lattices of small rank that every unimodular lattice of rank ≤ 22 has short roots, hence cannot arise from a smooth cubic fourfold.

We recall the notion of the discriminant form $q_A : A^\vee/A \to \mathbb{Z}/2\mathbb{Z}$ of A, as well as the modulo 8 signature $\operatorname{sign} q_A$ considered in [42, §1]. We remark that, in the notation of [42, Proposition 1.8.1], for any odd prime p, we have that $\operatorname{sign} q^p_\theta(p) \equiv 1 - p \bmod 4$ and $\operatorname{sign} q^p_\theta(p^2) \equiv 0 \bmod 8$ for any nonsquare class θ modulo p. For (2), by [42, Theorem 1.10.1], for a lattice A of odd rank ≤ 11 to be the intersection lattice of a smooth cubic fourfolds X, it is necessary that the signature satisfy $\operatorname{sign} q_A \equiv 11 - \rho \bmod 4$. If A has discriminant p, then $\operatorname{sign} q_A \equiv 1 - p \bmod 4$, hence we must have, $p \equiv \rho - 2 \bmod 4$. This is impossible if $p \equiv \rho \bmod 4$.

For (3), if 2 strictly divides $\operatorname{disc}(A)$, then $q_A = q^2_\theta(2) + q(\text{odd})$, where $q(\text{odd})$ means a finite quadratic form on a group of odd order. By [42, Prop. 1.11.2*], $\operatorname{sign} q^2_\theta(2) \not\equiv 0 \bmod 2$ while $\operatorname{sign} q(\text{odd}) \equiv 0 \bmod 2$. Hence no such cubic fourfold X exists. □

4. Clifford index bounds for cubic fourfolds

In this section, we recall the constructions of pfaffian cubic fourfolds by Beauville and Donagi [5] and Brill–Noether general K3 surfaces of degree 14 by Mukai [41, Thms. 3.9, 4.7], working up to a proofs of our main results. Throughout, denote by $\mathbb{P}(W)$ the projective space of lines in a vector space W.

Let V be a $\mathbb{C}$-vector space of dimension 6 and consider the subvarieties G and Δ of $\mathbb{P}(\bigwedge^2 V)$ of tensors of rank 2 and ≤ 4, respectively. Then G coincides with the image of the Plücker embedding $G(2, V) \hookrightarrow \mathbb{P}(\bigwedge^2 V)$, hence has dimension 8 and degree 14 by the Schubert calculus, see [13]. Also, Δ coincides with the vanishing locus of the pfaffian map $\mathrm{pf} : \bigwedge^2 V \to \bigwedge^6 V$, hence is a hypersurface of degree 3. We have that G is the singular locus of Δ and that Δ coincides with the secant variety of G, see [40, Rem. 1.5]. Similarly, we define $G^\vee \subset \Delta^\vee \subset P(\bigwedge^2 V^\vee)$. Here, $\mathbb{P}(\bigwedge^2 V^\vee)$ is the space of alternating bilinear forms on V up to homothety, and $\Delta^\vee$ is the subvariety of degenerate forms.

If $L \subset \mathbb{P}(\bigwedge^2 V)$ is a linear subspace of dimension 8 intersecting G transversally then $S = L \cap G \subset L \cong \mathbb{P}^8$ is the projective model of a smooth Brill–Noether general polarized K3 surface (S, H) of degree 14, see [41, Thm. 3.9]. Conversely, if (S, H) is a Brill–Noether general polarized K3 surface of degree 14, then S has a *rigid* vector bundle E, unique up to isomorphism, such that E is stable of rank 2 with $\det E \cong H$ and $\chi(S, E) = h^0(S, E) = 6$, see [41, Thm. 4.5]. In particular, the evaluation morphism $H^0(S, E) \otimes \mathscr{O}_S \to E$ is surjective, hence there is a grassmannian embedding $\Phi_E : S \to G(2, H^0(S, E)^\vee)$ taking $x \mapsto E^\vee_x$. Here, we think of $E^\vee_x \subset H^0(S, E)^\vee$ as a 2-dimensional subspace dual to the quotient map $H^0(S, E) \to E_x$ defined by the evaluation morphism. We have that $E = \Phi^*_E \mathscr{E}$, where $\mathscr{E}$ is the tautological rank 2 vector (sub)bundle on $G(2, H^0(S, E)^\vee)$. Composing with the Plücker embedding, we have an embedding $S \to \mathbb{P}(\bigwedge^2 H^0(S, E)^\vee)$. On the other hand, the exterior square $\bigwedge^2 H^0(S, E) \otimes \mathscr{O}_S \to \bigwedge^2 E$ of the evaluation morphism defines a linear map $\lambda : \bigwedge^2 H^0(S, E) \to H^0(S, \bigwedge^2 E) \cong H^0(S, H)$. As λ is surjective, we arrive at a commutative square of morphisms

$$\begin{tikzcd} S \arrow[r, "\Phi_E"] \arrow[d, "\Phi_H"'] & G(2, H^0(S,E)^\vee) \arrow[d, "\text{Plücker}"] \\ \mathbb{P}(H^0(S,H)^\vee) \arrow[r, "\mu"] & \mathbb{P}(\bigwedge^2 H^0(S,E)^\vee) \end{tikzcd}$$

where μ is the linear embedding defined by λ. A result of Mukai is that this square is cartesian, see [41, Thm. 4.7], hence S can be written as an intersection $S = L \cap G \subset \mathbb{P}(\bigwedge^2 V)$ in our previous notation, where $L = \mathbb{P}(H^0(S,H)^\vee)$, $V = H^0(S,E)^\vee$, and $G = G(2,V)$. In conclusion, a polarized K3 surface (S,H) of degree 14 is Brill–Noether general if and only if its projective model is a transversal intersection $S = G \cap L \subset \mathbb{P}(\bigwedge^2 V)$ and any such (S,H) has a rigid vector bundle of rank 2, i.e., a rank 2 stable vector bundle E with $\det(E) \cong H$ and $\chi(E) = 6$. By [39, Thm. 3.3(2)], Brill–Noether general polarized K3 surfaces $S = G \cap L$ and $S' = G \cap L'$ are projectively equivalent if and only if L and L' are equivalent under the action of $GL(V)$.

Still letting $L \subset \mathbb{P}(\bigwedge^2 V)$ be a linear subspace of dimension 8, if the projective dual linear subspace $L^\perp \subset \mathbb{P}(\bigwedge^2 V^\vee)$ of dimension 5 intersects $\Delta^\vee$ transversally, then $X = L^\perp \cap \Delta^\vee \subset L^\perp \cong \mathbb{P}^5$ is a pfaffian cubic fourfold, see [5, §2]. Conversely, writing $L^\perp = \mathbb{P}(W)$ for a subspace $W \subset \bigwedge^2 V^\vee$, then $L^\perp$ gives rise to a global section of the vector bundle $W^\vee \otimes \mathscr{O}_G(1)$, whose zero locus is precisely S.

By Hassett [19], any cubic fourfold of discriminant 14 has an associated K3 surface of degree 14.

Proposition 4.1. *Let V be a vector space of dimension 6 and let $L \subset \mathbb{P}(\bigwedge^2 V)$ be a linear subspace of dimension 8. Assume that $S = L \cap G$ has dimension 2 and that $X = L^\perp \cap \Delta^\vee$ has dimension 4. Then there is a semiorthogonal decomposition*

$$\mathsf{D}^b(X) = \langle \mathcal{A}_X, \mathscr{O}_X, \mathscr{O}_X(1), \mathscr{O}_X(2) \rangle$$

and an equivalence of categories $\mathcal{A}_X \cong \mathsf{D}^b(S)$. Furthermore, X is smooth if and only if S is smooth.

Proof. The semiorthogonal decomposition and equivalence of categories follows from Kuznetsov's theory of homological projective duality (cf. [29, Thm. 10.4] and also [31, Thm. 3.1]). By the existence of the semiorthogonal decomposition as well as Ext-boundedness considerations, the smoothness of X and of S is equivalent by [30, Lemmas 2.25, 2.26]. See also [32, Lemma 4.4]. □

To link pfaffian cubic fourfolds and curves of genus 8 on the associated K3, we will need the following.

Proposition 4.2. *A smooth cubic fourfold X is pfaffian if and only if it has a discriminant 14 marking whose associated K3 surface (S,H) is Brill–Noether general.*

Proof. First suppose that $X = L^\perp \cap \Delta^\vee$ is a smooth pfaffian cubic fourfold. By Proposition 4.1, $S = L \cap G$ is a K3 surface of degree 14 with a polarization H defined by the projective embedding $S \to L \cong \mathbb{P}^{14}$ and there is an equivalence $\mathcal{A}_X \cong \mathsf{D}^{\mathrm{b}}(S)$. By Mukai [39, Thms. 3.10], (S,H) is Brill–Noether general. By Addington–Thomas [1] (cf. [21, Prop. 3.3]), the equivalence $\mathcal{A}_X \cong \mathsf{D}^{\mathrm{b}}(S)$ induces a Hodge isometry of Mukai lattices $\widetilde{H}(\mathcal{A}_X, \mathbb{Z}) \cong \widetilde{H}(S, \mathbb{Z})$, which implies that X has a marking of discriminant 14 for which the associated polarized K3 surface is (S,H).

Now suppose that X is a smooth cubic fourfold with a marking of discriminant 14 whose associated polarized K3 surface (S,H) is Brill–Noether general. Then by Mukai [39, Prop. 4.7], H defines a projective embedding whose image is $S \cong L \cap G \subset L \cong \mathbb{P}^{14}$, for $L = \mathbb{P}(H^0(S,H)^\vee)$ as described above. Then $X' = L^\perp \cap \Delta^\vee \subset L^\perp \cong \mathbb{P}^5$ has expected dimension 4, hence by Proposition 4.1, X' is a smooth pfaffian cubic fourfold, $\mathsf{D}^{\mathrm{b}}(X')$ has a semiorthogonal decomposition $\langle \mathcal{A}_{X'}, \mathscr{O}_{X'}, \mathscr{O}_{X'}(1), \mathscr{O}_{X'}(2)\rangle$, and there is an equivalence $\mathcal{A}_{X'} \cong \mathsf{D}^{\mathrm{b}}(S)$. In particular, by Addington–Thomas [1], X' has a marking of discriminant 14 whose associated polarized K3 surface is (S, H). By the injectivity of $\mathcal{C}_{K_{14}} \hookrightarrow \mathcal{K}_{14}$, we have that X and X' are isomorphic. In particular, X is pfaffian. □

In terms of the open immersion of moduli spaces, Proposition 4.2 says that under the open immersion $\mathcal{C}_{K_{14}} \hookrightarrow \mathcal{K}_{14}$, the pfaffian locus coincides with the restriction of the Brill–Noether general locus to the image.

For the very general cubic fourfold X in $\mathcal{C}_{14}$, the (unique) associated polarized K3 surface (S, H) of degree 14 has Picard rank 1. The following is stated many times in the literature.

Corollary 4.3. *Any cubic fourfold X of discriminant 14 and with $A(X)$ of rank 2 is pfaffian.*

Proof. Since the associated K3 surface (S, H) has Picard rank 1, the smooth curves $C \in |H|$ are Brill–Noether general, by Lazarsfeld [34]. Hence Proposition 4.2 applies to show that X is pfaffian. □

As a further consequence, any cubic fourfold in $\mathcal{C}_{14} \smallsetminus \mathcal{P}f$ has $A(X)$ of rank ≥ 3.

By Proposition 4.2, X is not pfaffian if and only if (S, H) is Brill–Noether special for every degree 14 marking on X. Then by Theorem 1.7, we know that (S, H) must admit a lattice-polarization for some lattice in Table 1.

The main result of this section is the following.

Theorem 4.4. *Let X be a smooth cubic fourfold with a discriminant 14 marking and (S, H) an associated K3 surface of degree 14. Then $\gamma(S, H) = 3$.*

Proof. As noted in §1.2, if (S, H) is Brill–Noether general, then $\gamma(S, H) = 3$. So we can assume that (S, H) is Brill–Noether special. In particular, $A(X)$ has rank ≥ 3 and let $\Lambda \subset A(X)$ be a primitive sublattice of rank 3 containing the marking K_{14}. Then by Theorem 1.7, (S, H) would admit an appropriate $\sigma(\Lambda)$-polarization for $\sigma(\Lambda)$ given on Table 1. By Proposition 2.3, any such cubic fourfold X would have a Λ-polarization, for the unique rank 3 cubic fourfold lattice Λ with specified $\sigma(\Lambda)$. We have enumerated these lattices Λ in Table 1. We now show that each such lattice Λ corresponding to $\gamma(S, H) < 3$ has roots, hence no smooth Λ-polarized cubic fourfolds exist in these cases.

When $\gamma(S, H) = 0$, the cubic fourfold lattice Λ with $(b, c) = (6, 8)$ has short root $4h^2 - 3T + 2J$. Hence $A(X) \supset \Lambda$ would contain a short root, thus no such smooth cubic fourfold exists.

When $\gamma(S, H) = 1$, the cubic fourfold lattice Λ with $(b, c) = (5, 6)$ has long root $4h^2 - 3T + 3J$. Hence $A(X) \supset \Lambda$ would contain a long root, thus no such smooth cubic fourfold exists.

When $\gamma(S, H) = 2$, we have two choices. The cubic fourfold lattice Λ with $(b, c) = (2, 2)$ has short root J. Hence $A(X) \supset \Lambda$ would contain a short root, thus no such smooth cubic fourfold exists. The cubic fourfold lattice Λ with $(b, c) = (4, 4)$ has long root $4h^2 - 3T + 3J$. Hence $A(X) \supset \Lambda$ would contain a long root, thus no such smooth cubic fourfold exists.

This rules out all possibilities with $\gamma(S,H) < 3$. Hence (S,H) is Brill–Noether special with $\gamma(S,H) = 3$, and thus by Proposition 3.2, X admits a Π-polarization. $\square$

Remark 4.5. This is a continuation of Remark 1.9. The rank 3 cubic fourfold lattice Λ, whose associated K3 surface lattice $\sigma(\Lambda)$ has degree 14 generated by H and E with $E.H = 5$ and $E^2 = 0$, must have $(b,c) = (1,2)$. In this case, Λ has short root J. We conclude that there exist no smooth cubic fourfolds X whose associated K3 surface has a line bundle restricting to a g^1_5 on the smooth genus 8 curves in the polarization class.

Given an admissible $d = 2g - 2 > 6$, we wonder which values $0 \le \gamma \le \lfloor (g-1)/2 \rfloor$ can be realized by a Clifford indices polarized K3 surfaces associated to Brill–Noether special cubic fourfolds of discriminant d? For example, we expect that there are no "hyperelliptic" or "trigonal" special cubic fourfolds of any admissible discriminant.

5. Complement of the pfaffian locus

In this section, we can finally prove Theorem 1. We also show that cubic fourfolds can admit multiple discriminant 14 markings, some Brill–Noether special and some Brill–Noether general, implying that the pfaffian locus is not open inside $\mathcal{C}_{14}$.

Proof of Theorem 1. As a consequence of Theorem 4.4, the only component of the Brill–Noether special locus of $\mathcal{K}_{14}$ that intersects the image of $\mathcal{C}_{K_{14}}$ is the one corresponding to $\gamma = 3$ in Table 1. By Propositions 2.1 and 3.2, we have that the intersection of this component with the image of $\mathcal{C}_{K_{14}}$ in $\mathcal{K}_{14}$ coincides with the locus $\mathcal{C}_{\Pi} \subset \mathcal{C}_{K_{14}}$ (where we always consider Π with the fixed discriminant 14 marking derived from (1)). Applying the forgetful map, we see that the complement of the pfaffian locus in $\mathcal{C}_{14}$ is contained in $\mathcal{C}_{[\Pi]}$. $\square$

Example 5.1. Consider the lattice Λ:

$$\begin{array}{c|cccc} & h^2 & T & P & P' \\ \hline h^2 & 3 & 4 & 1 & 1 \\ T & 4 & 10 & 0 & 0 \\ P & 1 & 0 & 3 & 0 \\ P' & 1 & 0 & 0 & 3 \end{array}$$

One can check, using the result of Laza [33] and Looijenga [37] on the image of the period map, that Λ is a cubic fourfold lattice, implying that the locus $\mathcal{C}_{[\Lambda]}$ of cubic fourfolds admitting a Λ-polarized has codimension 3 in the moduli space $\mathcal{C}$. We remark that the explicit example found by computers in [6, Ex. A.2] is contained in $\mathcal{C}_{[\Lambda]}$, and motivated its definition. By Proposition 2.1, the classes P and P' correspond to disjoint planes, and the class T generates a marking of discriminant 14, hence $\mathcal{C}_{\Lambda}$ is a divisor in $\mathcal{C}_{\Pi}$. Consider the four discriminant 14 markings generated by the classes:

$$T, T' = 2h^2 - P - P', T'' = 3h^2 - T - P, T''' = 3h^2 - T - P'.$$

We compute that the polarized K3 surfaces (S, H) of degree 14 associated to these markings admit lattice-polarizations for the following lattices, respectively:

$$\begin{array}{c|ccc} & H & C & C' \\ \hline H & 14 & 2 & 2 \\ C & 2 & -2 & 1 \\ C' & 2 & 1 & -2 \end{array} \qquad \begin{array}{c|ccc} & H & L & C \\ \hline H & 14 & 7 & 4 \\ L & 7 & 2 & 2 \\ C & 4 & 2 & -2 \end{array} \qquad \begin{array}{c|ccc} & H & C & L \\ \hline H & 14 & 2 & 1 \\ C & 2 & -2 & 0 \\ L & 1 & 0 & -2 \end{array} \qquad \begin{array}{c|ccc} & H & C & L \\ \hline H & 14 & 2 & 1 \\ C & 2 & -2 & 0 \\ L & 1 & 0 & -2 \end{array}$$

Clearly T'' and T''' are permuted up to reordering the planes P and P', so the last two lattice polarizations are isomorphic.

Now assume that X is very general in $\mathcal{C}_{[\Lambda]}$. Then the associated K3 surfaces have Picard lattices isomorphic to the ones above and in all three cases, one can verify that the degree 14 polarization class is very ample by an analysis of the ample cone. In the first and third cases, one can also verify that the smooth genus 8 curves in the polarization class are Brill–Noether general, hence by Proposition 4.2, that X is pfaffian (in multiple ways). In the second case, one can verify that the second generator C restricts to a g^2_7 on the smooth genus 8 curves in the polarization class (hence they are Brill–Noether special by Lemma 1.2). By Theorem 1.3, the polarized K3 surface associated to this second marking is Brill–Noether special.

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