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CONIC BUNDLE FOURFOLDS WITH NONTRIVIAL UNRAMIFIED BRAUER GROUP

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Abstract

We derive a formula for the unramified Brauer group of a general class of rationally connected fourfolds birational to conic bundles over smooth threefolds. We produce new examples of conic bundles over $\mathbb{P}^3$ where this formula applies and which have nontrivial unramified Brauer group. The construction uses the theory of contact surfaces and, at least implicitly, matrix factorizations and symmetric arithmetic Cohen–Macaulay sheaves, as well as the geometry of special arrangements of rational curves in $\mathbb{P}^2$. We also prove the existence of universally CH_0-trivial resolutions for the general class of conic bundle fourfolds we consider. Using the degeneration method, we thus produce new families of rationally connected fourfolds whose very general member is not stably rational.

1. Introduction

One of the fundamental problems in the birational classification of algebraic varieties is to distinguish between varieties that are in some sense close to $\mathbb{P}^n$—e.g., stably rational, unirational, or rationally connected—and varieties in the birational equivalence class of $\mathbb{P}^n$ itself. Conic bundles over rational varieties are a natural class to study in this respect, and the literature on them is prodigious. For example, conic bundles over rational surfaces were used in [AM72] to produce varieties that are unirational but not stably rational (hence a fortiori not rational), and in [B-CT-S-SwD] to produce stably rational, but nonrational varieties. In [CT-O], the unramified cohomology groups were introduced to give a more systematic treatment of, and greatly generalize, the examples in [AM72]. There is also a whole body of work on conic bundles

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that are birationally rigid, taking its departure from the groundbreaking works [Sa80], [Sa82], [Is87]; see [Pukh13] for a survey.

Conic bundles are important from a deformation-theoretic perspective as well, as they usually come in families, making them amenable to the degeneration method introduced and developed in the seminal articles [Voi15] and [CT-P16]. The method relies on the ability to obstruct the universal triviality of the Chow group of 0-cycles on a mildly singular central fiber of such a family. Then the very general fiber of the family will be similarly obstructed, and in particular, will not be stably rational. The degeneration method has broadened the range of applicability of previously known obstructions such as unramified invariants and differential forms in positive characteristic, and notably, has very recently led to examples of families of smooth fourfolds with rational and nonrational fibers [HPT16].

The present article started from a close analysis of the example in [HPT16] of a quadric surface fibration over $\mathbb{P}^2$ with nontrivial unramified Brauer group, defined as divisor of bi-degree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$. While the projection to $\mathbb{P}^2$ gives the quadric surface fibration structure over $\mathbb{P}^2$, the other projection gives a conic bundle over $\mathbb{P}^3$. The structural features of this conic bundle helped us find the statements of the general results of Section 2 about the unramified Brauer group and of Section 6 about the singularities of conic bundles over threefolds. We also provide new constructions, in Sections 3, 4, and 5, of conic bundles where these results apply. One application is the following (see Theorem 6.6).

Theorem. *A very general conic bundle $Y \to \mathbb{P}^3$ over $\mathbb{C}$, defined by a homogeneous 3×3 matrix with entries of degrees*

$$\begin{pmatrix} 7 & 4 & 4 \\ 4 & 1 & 1 \\ 4 & 1 & 1 \end{pmatrix}$$

is not stably rational.

For further developments and more recent results on conic bundles with small discriminants see [ABB19], [ABB18], [BB17].

Let us describe the contents of the individual sections in more detail.

In Section 2, we provide a formula for the unramified Brauer groups of the total spaces of certain conic bundles over smooth projective threefolds B with $\mathrm{Br}(B)[2] = 0$ and $H^3_{\text{ét}}(B, \mathbb{Z}/2) = 0$ over an algebraically closed field k of characteristic not 2. The formula (given in Theorem 2.6) depends on the geometry and combinatorics of the components of the discriminant divisor and their mutual intersections, as well as the structure of their double covers induced by the lines in the fibers of the conic bundle. If the discriminant is irreducible, the unramified Brauer group of the conic bundle is trivial. The formula can be viewed as a higher dimensional analogue of a formula due to Colliot-Thélène (see [Pi16, Thm. 3.13]) for conic bundles over surfaces; see also [Zag77]. Such formulas are naturally stated in the language of Galois cohomology, algebraic K-theory, and Bloch–Ogus theory, but we go on to reinterpret ours in a geometric way in Corollary 2.9. This is fundamental for finding, in Sections 4 and 5, the geometric examples of conic bundles where the formula applies. In particular, we refer the reader to the roadmap at the start of Section 5 showing how we achieve this.

In Section 3, we introduce a method to produce fourfold conic bundles with reducible discriminants via taking double covers branched in surfaces that are contact to discriminants of simpler conic bundles. We analyze the example in [HPT16], of a divisor of bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$, as a conic bundle over $\mathbb{P}^3$ from this perspective, yielding an independent proof that this variety has nontrivial unramified Brauer group.

In Section 4, we introduce another method to construct fourfold conic bundles over $\mathbb{P}^3$ with reducible discriminants. It is again based on the theory of contact of surfaces developed largely in the fundamental paper [Cat81], as well as on the theory of matrix factorizations as in [Ei80] and the theory of symmetric determinantal representations of hypersurfaces [Cat81], [Beau00], [Dol12, Chapter 4]. While the latter two theoretical tools are not used logically in our proof, they were very important in finding the result.

In Section 5, we complete the construction of new examples of fourfold conic bundles over $\mathbb{P}^3$ with nontrivial unramified Brauer group. These are, hence, not stably rational. They are part of natural families of conic bundles of specific graded-free-types over $\mathbb{P}^3$.

Finally, in Section 6, we analyze the singularities of the total spaces of a quite general class of conic bundle fourfolds, proving that they admit universally CH_0-trivial resolutions. This is aided by a classification of local analytic normal forms for the singularities that can appear. The degeneration method of [Voi15] and [CT-P16] can then be applied to yield an obstruction to stable rationality of the very general member of families in which our new examples appear. In particular, this provides a simpler proof that the example considered in [HPT16] admits a universally CH_0-trivial resolution.

As a final note, it may be interesting to remark that we were only able to construct the examples in Sections 4 and 5 by translating virtually every algebraic concept entering in Theorem 2.6 into geometry. In this respect, hypersurfaces with symmetric rank 1 arithmetic Cohen–Macaulay sheaves are better than determinants, contact of surfaces is a more versatile concept than reducibility of polynomials, and special configurations of rational curves are more concrete than the analysis of functions becoming squares when restricted to a curve. On the other hand, the arithmetic function field and Galois cohomological point of view is far superior if one wants to prove an abstract general result such as Theorem 2.6. The main difficulty is then constructing examples. One reason why it is so much more difficult to find conic bundles over threefolds with prescribed discriminant, as opposed to over surfaces, is that the theory of maximal orders in quaternion algebras over threefolds is more complicated. Instead of relying on the theory of maximal orders, which was utilized in [AM72], we rely on geometry to construct our examples.

Conventions. The letter k will usually denote an algebraically closed ground field of characteristic not 2, unless explicitly stated otherwise. As usual, the term variety over k means a separated, integral scheme of finite-type over k. A conic bundle is a flat projective surjective morphism of varieties with (geometric) fibers isomorphic to plane conics and general fiber smooth.

2. Brauer group of conic bundles over threefolds

We first recall a few facts from Galois cohomology.

Let L be the function field of an integral variety Z defined over k. At this point we do not even have to assume that k is algebraically closed, but k should have characteristic different from 2. The first Galois cohomology group $H^1(L,\mathbb{Z}/2) := H^1(\mathrm{Gal}(L),\mathbb{Z}/2)$, with constant coefficients $\mathbb{Z}/2$, can be identified via Kummer theory with the group of square classes

$$H^1(L,\mathbb{Z}/2) \simeq L^\times/L^{\times 2}. \tag{1}$$

The second Galois cohomology group $H^2(L,\mathbb{Z}/2)$ can be identified with the 2-torsion subgroup of the Brauer group of L

$$H^2(L,\mathbb{Z}/2) \simeq \mathrm{Br}(L)[2]. \tag{2}$$

For $a, b \in L^\times$, we denote by the symbol $(a, b) \in \mathrm{Br}(L)[2]$ the Brauer class of the quaternion algebra generated by x, y with relations $x^2 = a$, $y^2 = b$, and $xy = -yx$. This is the same as the Brauer class associated to the plane conic over L defined by $ax^2 + by^2 = z^2$. It also coincides with the cup product of the square classes of a and b via the identification (1).

Now suppose D is a prime divisor of Z such that Z is regular in the generic point of D; thus D corresponds to a unique discrete divisorial valuation v_D of L with residue field $k(D)$. The two residue maps (homomorphisms) that will be relevant to us,

$$\begin{aligned} \partial_D^1 &\colon H^1(L,\mathbb{Z}/2) \to H^0(k(D),\mathbb{Z}/2) = \mathbb{Z}/2, \\ \partial_D^2 &\colon H^2(L,\mathbb{Z}/2) \to H^1(k(D),\mathbb{Z}/2), \end{aligned} \tag{3}$$

can be defined in the following manner: if a class in $H^1(L,\mathbb{Z}/2)$ is represented by an element $a \in L^\times$ according to (1), then $\partial_D^1(a) = v_D(a) \pmod 2$; if a class in $H^2(L,\mathbb{Z}/2)$ is represented by a symbol (a,b) according to (2), then

$$\partial_D^2(a,b) = (-1)^{v_D(a)v_D(b)}\overline{a^{v_D(b)}/b^{v_D(a)}}, \tag{4}$$

where $\overline{a^{v_D(b)}/b^{v_D(a)}} \in H^1(k(D),\mathbb{Z}/2) = k(D)^\times/k(D)^{\times 2}$ is the square class of the unit $a^{v_D(b)}/b^{v_D(a)} \in L^\times$ in the residue field. In fact ∂_D^2 is uniquely determined by the formula $\partial_D^2(\pi,u) = \overline{u}$ for any uniformizer π and unit u in the valuation ring of v_D. For $u \in L^\times$, we sometimes write $u|_D := \overline{u}$ for the residue class.

One also defines the map ∂_D^1 in the more general case when Z is potentially singular at the generic point of D, so that the local ring of Z at the generic point of D is not necessarily a discrete valuation ring. In that case, we define ∂_D^1 following Kato [Ka86, p. 151]. If $Z' \to Z$ is the normalization and $D_1,\ldots,D_\mu$ are the irreducible components lying over D corresponding to the discrete divisorial valuations of L with center D, then for $a \in L^\times$ we define

$$\partial_D^1(a) = \sum_{i=1}^{\mu} [k(D_i) : k(D)] v_{D_i}(a) \qquad \pmod 2. \tag{5}$$

The unramified cohomology group $H^2_{\mathrm{nr}}(L/k,\mathbb{Z}/2)$, which depends on the ground field k, is the subgroup of $H^2(L,\mathbb{Z}/2)$ consisting of those elements that are annihilated by all residue maps $\partial_v^2\colon H^2(L,\mathbb{Z}/2) \to H^1(\kappa(v),\mathbb{Z}/2)$ where v runs over the divisorial valuations of L that are trivial on k. Here $\kappa(v)$ is the residue field of v. Clearly, formula (4) makes sense for any divisorial valuation v of L, not only those v_D that have a divisorial center D on Z. The nontriviality of the unramified cohomology group is an obstruction to stable rationality of L over k.

If Z is smooth and proper over k, then there is a natural isomorphism $\mathrm{Br}(Z)[2] \to H^2_{\mathrm{nr}}(L/k,\mathbb{Z}/2)$, where $\mathrm{Br}(Z) = H^2_{\text{ét}}(Z,\mathbb{G}_m)$ is the cohomological Brauer group of Z, cf. [CT95, Prop. 4.2.3(a)]. In general, we refer to $H^2_{\mathrm{nr}}(L/k,\mathbb{Z}/2)$ as the 2-torsion in the unramified Brauer group, and write it as $\mathrm{Br}_{\mathrm{nr}}(L/k)[2]$.

In practice one uses complementary results to narrow down the set of divisorial valuations required to check in the definition of unramified cohomology to those corresponding to prime divisors on a *fixed* model of L. Such results are implied by so-called "purit" [CT95] and we will use a variant of [CT95, Thm. 3.82]; see also [Pi16, Prop. 3.2].

Proposition 2.1. *Let $\mathscr{O}$ be the local ring of a smooth (scheme-theoretic) point on a variety over a field k of characteristic not 2, and let L be the field of fractions of $\mathscr{O}$. Let $\gamma \in H^i(L, \mathbb{Z}/2)$ be some class such that $\partial^i_v(\gamma) = 0$ for all valuations corresponding to height one prime ideals of $\mathscr{O}$ (hence prime divisors in $\mathrm{Spec}(\mathscr{O})$). Then γ is in the image of the natural map*

$$H^i_{\text{ét}}(\mathrm{Spec}(\mathscr{O}), \mathbb{Z}/2) \to H^i(L, \mathbb{Z}/2).$$

The following corollary, which employs an argument due to Bloch and Ogus [BO74], is a little more geometric; cf. [CT95, Prop. 2.1.8(d)].

Corollary 2.2. *Suppose Z_{sm} is a smooth integral variety over a field k of characteristic not 2, and let L be the function field of Z_{sm}. Then every element in $H^i(L/k, \mathbb{Z}/2)$ that is unramified with respect to divisorial valuations corresponding to prime divisors on Z_{sm} is also unramified with respect to all divisorial valuations that have centers on Z_{sm}.*

We will often apply the corollary above to the smooth locus $Z_{\mathrm{sm}} := Z \backslash Z_{\mathrm{sing}}$ of a proper variety Z over k, where Z_{sing} is its singular locus.

Let K be an arbitrary field (possibly of characteristic 2) and let C be a smooth projective curve of genus zero over K. The anticanonical class on C defines an embedding $C \to \mathbb{P}^2_K$ as a smooth plane conic; we call C a smooth conic over K. As remarked earlier, a smooth conic C determines a Brauer class $\alpha \in \mathrm{Br}(k)[2]$. We say that C is nonsplit if $C(K) = \emptyset$, equivalently, α is nontrivial. As before, we set $\mathrm{Br}(C) := H^2_{\text{ét}}(C, \mathbb{G}_m)$. Since $\mathrm{Br}(K) = H^2(K, \mathbb{G}_m) = H^2_{\text{ét}}(\mathrm{Spec}\, K, \mathbb{G}_m)$ for any field K, we have a pullback map $\mathrm{Br}(K) \xrightarrow{\iota} \mathrm{Br}(C)$. We will need the following.

Lemma 2.3. *Let C be a smooth nonsplit conic over an arbitrary field K. Then the pullback map induces an exact sequence*

$$0 \to \mathbb{Z}/2 \to \mathrm{Br}(K) \xrightarrow{\iota} \mathrm{Br}(C) \to 0 \tag{6}$$

where the kernel is generated by the Brauer class $\alpha \in \mathrm{Br}(K)[2]$ determined by C.

Assuming that K has characteristic not 2 and that -1 is a square, then (6) restricts to an exact sequence

$$0 \to \mathbb{Z}/2 \to \mathrm{Br}(K)[2] \to \mathrm{Br}(C)[2] \to \mathbb{Z}/2 \to 0, \tag{7}$$

and any class in $\mathrm{Br}(C)[2]$ is contained in the image of $\mathrm{Br}(K)[4] \to \mathrm{Br}(C)[4]$.

Proof. The proof of (6) is well known, but we summarize it here for convenience, cf. [CT-O, Prop. 1.5]. The identification of the kernel of ι is due to Witt [Wit35], and follows from the fact that C is a Severi–Brauer variety associated to the Brauer class α. The proof of the surjectivity of ι follows an argument with the Hochschild–Serre spectral sequence going back to the work of Lichtenbaum [Lic69], Iskovskikh, and Manin. We recall this argument here for convenience. Let K^s be a separable closure of K and let Γ be the Galois group of K^s/K. The exact sequence of low degree terms of the Hochschild–Serre spectral sequence and Hilbert's theorem 90 gives

$$\begin{aligned}0 \to \operatorname{Pic}(C) \to \operatorname{Pic}(C_{K^s})^{\Gamma} \to \operatorname{Br}(K) \to \ker\bigl(\operatorname{Br}(C) \to \operatorname{Br}(C_{K^s})\bigr)\\ \to H^1(\Gamma, \operatorname{Pic}(C_{K^s})).\end{aligned}$$

Since C is a smooth conic, it has a separable splitting field by [BrauerIII, Cor. 1.3], hence $C_{K^s} \cong \mathbb{P}^1_{K^s}$. For the vanishing of $\operatorname{Br}(\mathbb{P}^1_{K^s})$, one can appeal to (a generalization of) Tsen's theorem on the vanishing of the Brauer group of the function field of a curve over a separably closed field. We also use the fact that $\operatorname{Pic}(\mathbb{P}^1_{K^s}) = \mathbb{Z}$ has trivial Galois action and $H^1(\Gamma, \mathbb{Z}) = 0$, while $\operatorname{Pic}(C)$ is generated by $\omega_C^\vee$, which has degree 2, when C is a nonsplit conic. Hence the above sequence of low degree terms collapses to the desired exact sequence.

As for the second part, the fact that any element of $\operatorname{Br}(C)[2]$ is in the image of $\operatorname{Br}(K)[4] \to \operatorname{Br}(C)[4]$ follows immediately from (6), since the kernel has order 2. For the calculation of the cokernel of ι, the short exact sequence of group schemes $1 \to \mu_2 \to \mu_4 \to \mu_2 \to 1$ (assuming that K has characteristic not 2) induces a long exact sequence in Galois cohomology

$$\begin{aligned}\cdots \to H^1(K, \mu_2) \to H^2(K, \mu_2) \to H^2(K, \mu_4) \to H^2(K, \mu_2)\\ \to H^2(K, \mu_2) \to \cdots\end{aligned}$$

where the boundary maps are given by cup product with the class $(-1) \in H^1(K, \mathbb{Z}/2)$; cf. [Kah89, Lemmas 1,2]. Hence all boundary maps are zero if -1 is a square in K. Since K has characteristic not 2, we have $\operatorname{Br}(K)[n] = H^2(K, \mu_n)$ for n a power of 2. We then have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Br}(K)[2] \arrow[r] \arrow[d] & \operatorname{Br}(K)[4] \arrow[r] \arrow[d] & \operatorname{Br}(K)[2] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Br}(C)[2] \arrow[r] & \operatorname{Br}(C)[4] \arrow[r] & \operatorname{Br}(C)[2] &
\end{tikzcd}$$

and the snake lemma yields that

$$\operatorname{coker}\bigl(\operatorname{Br}(K)[2] \to \operatorname{Br}(C)[2]\bigr) \cong \ker\bigl(\operatorname{Br}(K)[2] \to \operatorname{Br}(C)[2]\bigr) \cong \mathbb{Z}/2$$

as desired; cf. [KRS98, §7]. We use the fact that $\mathrm{Br}(K)[4] \to \mathrm{Br}(C)[4]$ maps onto $\mathrm{Br}(C)[2]$ to see that the map

$$\mathrm{coker}\big(\mathrm{Br}(K)[2] \to \mathrm{Br}(C)[2]\big) \to \mathrm{coker}\big(\mathrm{Br}(K)[4] \to \mathrm{Br}(C)[4]\big)$$

is zero, even though $\mathrm{coker}\big(\mathrm{Br}(K)[4] \to \mathrm{Br}(C)[4]\big)$ might itself be nonzero. ☐

Definition 2.4. Let $\pi \colon Y \to B$ be a conic bundle over a smooth projective threefold B over an algebraically closed ground field k of characteristic not 2. Let S be its discriminant locus with its natural determinantal scheme structure. Let $S_1, \dots, S_n$ be its irreducible components.

We call the discriminant locus S *good* if S is reduced and if for each i, the fiber Y_s for general $s \in S_i$ consists of two distinct lines, and the natural double covers $\tilde{S}_i \to S_i$ determined by π in that case are irreducible.

Remark 2.5. Keeping the notation of the previous definition, if S is good and $\alpha \in H^2(K, \mathbb{Z}/2)$ is the Brauer class corresponding to the generic fiber of π, then the surfaces S_i are precisely those surfaces $\Sigma \subset B$ such that $\partial^2_\Sigma(\alpha) \neq 0$. If we drop the assumption that the cover $\tilde{S}_i \to S_i$ be irreducible, then we could get a trivial class in $H^1(k(S_i), \mathbb{Z}/2) = k(S_i)^\times / k(S_i)^{\times 2}$.

We can now go back to our geometric situation and state an algebraic version of the theorem that computes $H^2_{\mathrm{nr}}(k(Z)/k, \mathbb{Z}/2)$ for us in many cases.

Theorem 2.6. *Let k be an algebraically closed field of characteristic not 2 and let $\pi \colon Y \to B$ be a conic bundle over a smooth projective threefold B over k. Let $\alpha \in \mathrm{Br}(K)[2]$ be the Brauer class in $K = k(B)$ corresponding to the generic fiber of π. Assume that the discriminant locus of π is good with components $S_1, \dots, S_n$. We will also assume the following:*

a) *The vanishing $\mathrm{Br}(B)[2] = 0$ and $H^3_{\text{ét}}(B, \mathbb{Z}/2) = 0$ holds.*
b) *Through any irreducible curve in B, there pass at most two surfaces from the set $S_1, \dots, S_n$.*
c) *Through any point of B, there pass at most three surfaces from the set $S_1, \dots, S_n$.*
d) *For all $i \neq j$, S_i and S_j are factorial at every point of $S_i \cap S_j$.*

Put

$$\gamma_i = \partial^2_{S_i}(\alpha) \in H^1(k(S_i), \mathbb{Z}/2).$$

Define a subgroup Γ of the group $\bigoplus_{i=1}^n H^1(k(S_i), \mathbb{Z}/2)$ by

$$\Gamma = \bigoplus_{i=1}^{n} \langle \gamma_i \rangle.$$

Thus $\Gamma \simeq (\mathbb{Z}/2)^n$. We will write elements of Γ as $(x_1, \dots, x_n)$ with $x_i \in \{0, 1\}$.

Let $H \subset \Gamma$ consist of those elements $(x_1, \dots, x_n) \in (\mathbb{Z}/2)^n$ such that $x_i = x_j$ for $i \neq j$ whenever there exists an irreducible component C of $S_i \cap S_j$ such that either

- i) $\partial^1_C(\gamma_i) = \partial^1_C(\gamma_j) = 1$ *or*
- ii) $\partial^1_C(\gamma_i) = \partial^1_C(\gamma_j) = 0$, *and* $\gamma_i|_C$ *and* $\gamma_j|_C$ *are not both zero in* $H^1(k(C), \mathbb{Z}/2)$.

Then the 2*-torsion of the unramified Brauer group* $H^2_{\mathrm{nr}}(k(Y)/k, \mathbb{Z}/2)$ *of* Y *contains the subquotient* $H/\langle(1, \dots, 1)\rangle$ *by the "diagonal subgroup"* $\langle(1, \dots, 1)\rangle$ *of* Γ*, and is equal to it under the following additional geometric assumption*

- iii) *If* $\partial^1_C(\gamma_i) = \partial^1_C(\gamma_j) = 0$ *for some irreducible component* C *of the intersection* $S_i \cap S_j$*, then* S_i *and* S_j *intersect generically transversally along* C *and the rank of the conics in the fibers of* Y *is generically* 2 *over* C.

Later, we will reformulate various portions of Theorem 2.6 more geometrically. Before embarking on the proof, a few explanatory remarks are in order.

Remark 2.7. We do not know if the assumption iii) is necessary or redundant, i.e., whether we have equality $H^2_{\mathrm{nr}}(k(Y)/k, \mathbb{Z}/2) = H/\langle(1, \dots, 1)\rangle$ without it. It is conceivable that in any case there is a conic bundle $Y' \to B$, birational to Y over B, such that iii) is satisfied. However, for us iii) serves as a harmless simplifying assumption.

Remark 2.8. Conditions b) and c) are obviously simplifying assumptions on the intersection graph of the $S_1, \dots, S_n$. They could be replaced by different ones, but this would make the description of the unramified Brauer group $H^2_{\mathrm{nr}}(k(Y)/k, \mathbb{Z}/2)$ messier. On the other hand, condition d) is a hypothesis on the local algebraic structure, and something of that sort is probably indispensable in any version of Theorem 2.6. Condition a) is needed to glue certain Galois H^1-classes into Brauer classes on B as we will see below.

Proof of Theorem 2.6. It is a bit lengthy and we divide it into steps to make the logic clearer.

Step 1. *Inducing all potentially unramified Brauer classes in* $H^2_{\mathrm{nr}}(k(Y)/k, \mathbb{Z}/2)$ *from Brauer classes on* B *that are glued from a compatible set of* $\gamma_i = \partial^2_{S_i}(\alpha)$. The first question is how we can describe a totality of classes in $H^2(k(Y), \mathbb{Z}/2)$ that are the only candidates to yield unramified classes in $H^2_{\mathrm{nr}}(k(Y)/k, \mathbb{Z}/2)$. This is done via the following commutative diagram:

(8)

$$\begin{tikzcd}
 & & 0 & & \\
 & & \mathbb{Z}/2 \arrow[u] & & \\
0 \arrow[r] & H^2_{\mathrm{nr}}(k(Y)/Y, \mathbb{Z}/2) \arrow[r] & H^2_{\mathrm{nr}}(k(Y)/K, \mathbb{Z}/2) \arrow[u] \arrow[r, "\oplus \partial^2_T"] & \bigoplus_{T \in Y_B^{(1)}} H^1(k(T), \mathbb{Z}/2) & \\
 & \mathrm{Br}_{\mathrm{nr}}(K)[2] = 0 \arrow[r] & H^2(K, \mathbb{Z}/2) = \mathrm{Br}(K)[2] \arrow[u, "\iota"] \arrow[r, "\oplus \partial^2_S"] & \bigoplus_{S \in B^{(1)}} H^1(k(S), \mathbb{Z}/2) \arrow[u, "\tau"] \arrow[r, "\oplus \partial^1_C"] & \bigoplus_{C \in B^{(2)}} H^0(k(C), \mathbb{Z}/2) \\
 & & \langle \alpha \rangle \arrow[u] & \mathscr{K} \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

We will start by explaining the new pieces of notation: $H^2_{\mathrm{nr}}(k(Y)/Y, \mathbb{Z}/2)$ denotes all those classes in $H^2(k(Y), \mathbb{Z}/2)$ which are unramified with respect to divisorial valuations corresponding to prime divisors (threefolds) on Y. Note that the singular locus of Y has codimension ≥ 2 by our assumptions. By Corollary 2.2, we can also characterize $H^2_{\mathrm{nr}}(k(Y)/Y, \mathbb{Z}/2)$ as all those classes in $H^2(k(Y), \mathbb{Z}/2)$ that are unramified with respect to divisorial valuations which have centers on Y which are not contained in Y_{sing}. Moreover, $H^2_{\mathrm{nr}}(k(Y)/K, \mathbb{Z}/2)$ is the subset of those classes in $H^2(k(Y), \mathbb{Z}/2)$ which are unramified with respect to divisorial valuations that are trivial on K, hence correspond to prime divisors of Y dominating the base B (since π is of relative dimension 1).

In the upper row, T runs over all irreducible threefolds, i.e., prime divisors, in Y that do not dominate the base B, hence map to some surface in B. We call this set of irreducible threefolds $Y_B^{(1)}$. Then the upper row is exact by the very definitions.

In the lower row, S runs over the set of all irreducible surfaces $B^{(1)}$ in B and C over the set of all irreducible curves $B^{(2)}$ in B. Thus this row coincides with the usual Bloch–Ogus complex for degree 2 étale cohomology associated to B. The ith cohomology group of this complex is computed by the Zariski cohomology $H^i(B, \mathscr{H}^2)$ of the étale cohomology sheaf $\mathscr{H}^i$, which is the sheafification of the Zariski presheaf $U \mapsto H^2_{\text{ét}}(U, \mathbb{Z}/2\mathbb{Z})$; see [BO74, Thm. 6.1]. In particular, the lower row is exact in the first two places because $H^0(B, \mathscr{H}^2) = \mathrm{Br}(B)[2] = 0$ and $H^1(B, \mathscr{H}^2) \subset H^3_{\text{ét}}(B, \mathbb{Z}/2) = 0$ by hypothesis, where the later inclusion arises from the sequence of low terms associated to the Bloch–Ogus spectral sequence $H^i(B,\mathscr{H}^j) \Rightarrow H^{i+j}_{\text{ét}}(B,\mathbb{Z}/2)$; cf. [Kah95, §1.1].

Now let us discuss the vertical arrows. The left vertical column is Lemma 2.3. The map τ, defined by pullback under the field extensions $k(T) \supset k(S)$, coincides with the induced $k(S)^\times/k(S)^{\times 2} \to k(T)^\times/k(T)^{\times 2}$. If the generic fiber of $T \to S$ is geometrically integral, then $k(S)$ is algebraically closed inside $k(T)$, hence this induced map is injective. This is the case if S is not contained in the discriminant locus, since then the generic fiber of $T \to S$ is a smooth conic. If $S = S_i$ is a component of the discriminant locus, then the generic fiber of $T_i \to S_i$ is geometrically the union of two lines; Stein factorization displays this generic fiber as a line over the quadratic extension $F/k(S_i)$ defined by the residue class $\gamma_i \in H^1(k(S_i),\mathbb{Z}/2)$. In this case, the restriction-corestriction exact sequence in Galois cohomology implies that the kernel of the natural map $H^1(k(S_i),\mathbb{Z}/2) \to H^1(F,\mathbb{Z}/2)$ is generated by γ_i (and also the natural map $H^1(F,\mathbb{Z}/2) \to H^1(F(t),\mathbb{Z}/2)$ is injective). We conclude that the kernel of τ is

$$\mathscr{K} \simeq \langle\gamma_1\rangle \oplus \cdots \oplus \langle\gamma_n\rangle = \Gamma. \tag{9}$$

We argue that even though ι is not surjective, the subgroup

$$H^2_{\mathrm{nr}}(k(Y)/Y,\mathbb{Z}/2) \subset H^2_{\mathrm{nr}}(k(Y)/K,\mathbb{Z}/2)$$

is in the image of ι. By Lemma 2.3, any element

$$\zeta \in H^2_{\mathrm{nr}}(k(Y)/K,\mathbb{Z}/2)$$

not in the image of ι lifts to some $\xi \in H^2(K,\mathbb{Z}/4)$ of order 4. Then at least one residue $\partial^2_S(\xi) \in H^1(k(S),\mathbb{Z}/4)$ must have order 4, since the map $\oplus\partial^2_S$ is injective (i.e., we consider the lower row of diagram (8) with $\mathbb{Z}/4$ coefficients now). Since $\mathscr{K}$ is an elementary abelian 2-group and also equals the kernel of the map τ for $\mathbb{Z}/4$ coefficients

$$\tau\colon \bigoplus_{S\in B^{(1)}} H^1(k(S),\mathbb{Z}/4) \to \bigoplus_{T\in Y_B^{(1)}} H^1(k(T),\mathbb{Z}/4),$$

$\tau(\partial^2_S(\xi)) \in H^1(k(T),\mathbb{Z}/4)$ cannot be trivial. Since the diagram commutes, we see that $\partial^2_T(\zeta)$ is nontrivial, hence ζ cannot lie in $H^2_{\mathrm{nr}}(k(Y)/Y,\mathbb{Z}/2)$. This same diagram chase for the diagram (8) yields that the group $H^2_{\mathrm{nr}}(k(Y)/Y,\mathbb{Z}/2)$ can be described as the quotient by $\langle(1,\dots,1)\rangle$ of the subgroup $H' \subset (\mathbb{Z}/2)^n$ defined only using condition i) of the definition of H in the statement of Theorem 2.6. Note also that we use assumption b) (namely, each C determines a unique pair S_i, S_j such that C is a component of $S_i \cap S_j$) to ensure that elements in H' make up the kernel of $\oplus(\oplus\partial^1_C)$ in diagram (8).

Step 2. *Figuring out which classes in H' give classes in $H^2_{\mathrm{nr}}(k(Y)/k, \mathbb{Z}/2)$ by checking whether they are unramified with respect to all divisorial valuations ν of $k(Y)$: A case-by-case analysis depending on the dimension and location of the center of ν on B.*

We pick a class $\beta \in H^2(K, \mathbb{Z}/2)$ corresponding to an element in H', and denote by β' the image of β in $H^2(k(Y), \mathbb{Z}/2)$. We want to show that β' is unramified on Y if and only if β is in H. We first prove the if part by a case-by-case analysis, and the only if part in Step 3 below.

Step 2(a). *The center of ν on B is not contained in the intersection of two or more of the discriminant components.* Denote by $\mathscr{O}$ the local ring of the center Z of ν on B. Then $\beta - \alpha$ is in the image of $H^2_{\text{ét}}(\mathscr{O}, \mathbb{Z}/2)$ by Proposition 2.1. But $\iota(\beta - \alpha) = \iota(\beta)$, so this class is also unramified with respect to ν in this case.

Step 2(b). *The center ν on B is a curve C that is an irreducible component of $S_i \cap S_j$.*

Let $\mathscr{O}$ be the local ring of C in B. If β has $x_i = x_j = 1$, then again $\beta - \alpha$ is in the image of $H^2_{\text{ét}}(\mathscr{O}, \mathbb{Z}/2)$ by Proposition 2.1, and we conclude as before. So we can assume $x_i = 1, x_j = 0$ and then also $\partial^1_C(\gamma_i) = 0$. This condition means that a function representing $\gamma_i = \partial^2_{S_i}(\beta) \in H^1(k(S_i), \mathbb{Z}/2) = k(S_i)^\times/k(S_i)^{\times 2}$ has a zero or pole of even order along C. Moreover, γ_j can be represented by 1 in $k(S_j)^\times$. Passing to the inverse of the function representing γ_i if necessary (multiplying by squares does not change its class in $H^1(k(S_i), \mathbb{Z}/2)$), we can assume that it is contained in the local ring $\mathscr{O}_{S_i,C}$ of C in S_i. Call this function f_{γ_i}. Choose a local equation t for C in $\mathscr{O}_{S_i,C}$. Note that S_i is factorial along C, so C is a Cartier divisor on S_i.

Then $f_{\gamma_i}/(t^{v_C(f_{\gamma_i})})$ is a unit in $\mathscr{O}_{S_i,C}$, hence any preimage in $\mathscr{O}$ will be a unit. Call this preimage u_{γ_i}. For u_{γ_j} we could take 1. Now viewing u_{γ_i} as a rational function in K, the function field of B, and choosing a local equation π_{S_i} for S_i in $\mathscr{O}$ (also viewed as a function in K) we can form the symbol $(u_{\gamma_i}, \pi_{S_i}) \in H^2(K, \mathbb{Z}/2)$. Using formula (4), we conclude that

$$\partial^2_{S_i}(\beta) = \gamma_i = \partial^2_{S_i}(u_{\gamma_i}, \pi_{S_i})$$

by construction of u_{γ_i}. Moreover, $\beta - (u_{\gamma_i}, \pi_{S_i})$ is then in the image of $H^2_{\text{ét}}(\mathscr{O}, \mathbb{Z}/2)$ using Proposition 2.1 again. Here we are using that we have lifted f_{γ_i} to a unit u_{γ_i} to ensure that $\partial^2_S(u_{\gamma_i}, \pi_{S_i}) = 0$ for every other surface S different from S_i through C. Hence

$$\partial^2_\nu(\iota(\beta - (u_{\gamma_i}, \pi_{S_i}))) = 0,$$

so we will have shown that $\partial^2_\nu(\iota(\beta)) = \partial^2_\nu(\beta') = 0$ once we know $\partial^2_\nu(\iota(u_{\gamma_i}, \pi_{S_i})) = 0$. By formula (4) we have (up to a sign)

$$\partial^2_\nu(\iota(u_{\gamma_i}, \pi_{S_i})) = \overline{u_{\gamma_i}^{\nu(\pi_{S_i})}/\pi_{S_i}^{\nu(u_{\gamma_i})}} = \overline{u_{\gamma_i}^{\nu(\pi_{S_i})}} \in H^1(\kappa(\nu), \mathbb{Z}/2), \tag{10}$$

where the second equality follows because u_{γ_i} is a unit along C; note that here we are viewing all rational functions in K as functions in $k(Y)$ via the natural extension $K \subset k(Y)$.

On the other hand, (up to a sign)

$$\partial^2_C(\gamma_i, t) = \overline{\gamma_i^{\nu_C(t)}/t^{\nu_C(\gamma_i)}} = \overline{u_{\gamma_i}|_C} \in H^1(k(C), \mathbb{Z}/2), \tag{11}$$

where the second equality follows because f_{γ_i} and the function $u_{\gamma_i}|_{S_i}$ on S_i differ by a square, by construction.

But since the term in (11) is zero by assumption, so is the term in formula (10).

Step 2(c). *The center of ν is a point $p \in C$ as in Step* 2(b)*, and S_i, S_j are the only surfaces among the $S_1, \dots, S_n$ passing through p.*

Let $\mathscr{O}$ denote the local ring of p in B. If $x_i = x_j = 1$ we conclude as above by looking at $\beta - \alpha$. So assume $x_i = 1, x_j = 0$. Then $\partial^1_C(\gamma_i) = 0$. Note that we can find a local equation t for C in $\mathscr{O}_{S_i,p}$ since C is Cartier by the hypothesis that S_i is factorial along C. Pick a function $f_{\gamma_i} \in k(S_i)$ representing γ_i. Moreover, for any other irreducible curve C' passing through p, either in $S_i \cap S_j$ or lying entirely on S_i or S_j, we will have $\partial^1_{C'}(\gamma_i) = 0$, too. Let $C_1, \dots, C_N$ be all irreducible curves through p along which f_{γ_i} has a zero or pole, and pick a local equation t_ι in $\mathscr{O}_{S_i,p}$ for every C_ι. The rational function $f_{\gamma_i}/\{t_1^{v_{C_1}(f_{\gamma_i})} \cdots t_N^{v_{C_N}(f_{\gamma_i})}\}$ on S_i does not vanish or have a pole on any curve on S_i that passes through p. Hence, since S is assumed to be factorial, in particular, normal in p, this function is a unit locally around p, and can be lifted to a unit in $\mathscr{O}$. We call this u_{γ_i} again. Repeating the rest of the proof in Step 2(b) verbatim, with $k(C)$ replaced by $k(P)$, and using that every element in $k(P)$ is a square since k is algebraically closed, we see that $\partial_\nu(\beta') = 0$ here as well.

Step 2(d). *The center of ν is a point p that lies on exactly three surfaces S_i, S_j, S_k.*

Then $p \in S_i \cap S_j \cap S_k$. If we have $x_i = x_j = x_k = 1$, we can again pass to $\beta - \alpha$ and argue as above, so we can assume $x_i = 1, x_j = x_k = 0$, or $x_i = 0, x_j = x_k = 1$. Moreover, without loss of generality, we can assume β is of type $x_i = 1, x_j = x_k = 0$ since if it is of type $x_i = 0, x_j = x_k = 1$, $\beta - \alpha$ will be of type $x_i = 1, x_j = x_k = 0$, and $\partial_\nu(\iota(\beta - \alpha)) = \partial_\nu(\iota(\beta))$. Let $\mathscr{O}$ be the local ring of p in B again. Since every curve C on S_i passing through p, either on $S_i \cap S_j$ or $S_i \cap S_k$, or only on S_i, is Cartier on the surface S_i, we can find a unit u_{γ_i} in $\mathscr{O}$ that, when restricted to S_i, has the same class as γ_i in $H^1(k(S_i), \mathbb{Z}/2)$. We just repeat the argument in Step 2(b). The rest of the argument is then verbatim as in Step 2(b) (or Step 2(c)) with $k(C)$ again replaced by $k(P)$.

Step 3. *Proving that a class β in H' yields an unramified class β' on Y only if $\beta \in H$.*

We have to prove that if β has $x_i = 1$ and $x_j = 0$, so that $\partial^1_C(\gamma_i) = \partial^1_C(\gamma_j) = 0$ for every irreducible component C of $S_i \cap S_j$, and if $\gamma_i|_C$ and $\gamma_j|_C$ are nonzero in $H^1(k(C), \mathbb{Z}/2)$, then β' is ramified with respect to some divisorial valuation ν of $k(Y)$.

We now make use of assumption iii). Because of this, a local calculation, done later in Proposition 6.7, shows the following: there is a unique irreducible curve C' in which Y is singular and which dominates C in this case. Also, the map $C' \to C$ is generically one-to-one. Moreover, blowing up Y in C' yields an exceptional divisor E that is generically a $\mathbb{P}^1 \times \mathbb{P}^1$ bundle over C', hence birational to $\mathbb{P}^1 \times \mathbb{P}^1 \times C'$. Let $\nu = \nu_E$ be the associated valuation. Looking back at the computations in Step 2 above, and keeping the notation there, we see from formula (10) and the fact that $\nu_E(\pi_{S_i}) = 1$ (again a local calculation) that $\partial^2_\nu(\beta')$ is equal to $\bar{u}_{\gamma_i}$, viewed as an element of $H^1(k(E), \mathbb{Z}/2)$. Hence, this is nothing but the image, under the natural map $H^1(k(C), \mathbb{Z}/2) \to H^1(k(E), \mathbb{Z}/2)$, of $\bar{u}_{\gamma_i}$, viewed as an element of $H^1(k(C), \mathbb{Z}/2)$. But a nonsquare in a field cannot become a square in a purely transcendental extension of that field, hence $\partial^2_\nu(\beta') \neq 0$ in this case. ☐

We can reformulate parts of Theorem 2.6 to obtain the following geometric corollary that gives sufficient conditions for a conic bundle $\pi\colon Y \to B$ to have nontrivial $H^2_{\mathrm{nr}}(k(Y)/k, \mathbb{Z}/2)$.

Corollary 2.9. *Let k be again some algebraically closed ground field of characteristic not equal to 2, $\pi\colon Y \to B$ a conic bundle over a smooth projective threefold B with $\mathrm{Br}(B)[2] = H^3_{\text{ét}}(B, \mathbb{Z}/2) = 0$.*

Suppose that the discriminant locus $S = \bigcup_{i=1}^n S_i$ of π is good and $n \geq 2$ and suppose that assumptions b), c), d) *in Theorem* 2.6 *are satisfied.*

Suppose that for all $i \neq j$ and every irreducible component C of $S_i \cap S_j$, the fibers of π over a general point of C are still two distinct lines, and that the corresponding double cover $\tilde{C} \to C$ (inside $\tilde{S}_i$ or $\tilde{S}_j$) is reducible.

Then the unramified Brauer group of Y is nontrivial.

Proof. The fact that the fibers of π over a general point of C are still two distinct lines means $\partial^1_C(\gamma_i) = \partial^1_C(\gamma_j) = 0$. The condition that $\tilde{C}$ is reducible means that $\partial^2_C(\gamma_i, u)$ and $\partial^2_C(\gamma_j, u)$ are zero. ☐

3. Reducibility of the discriminant: 1st method

Subsequently, we will usually restrict our attention to conic bundles of *graded-free-type* over $\mathbb{P}^3$, informally, those defined by a graded symmetric 3×3 matrix. We now make this precise.

Definition 3.1. Fix a triple of nonnegative integers

$$(d_1, d_2, d_3) \in \mathbb{N}^3 \quad \text{such that } d_i \equiv d_j \pmod 2 \ \forall\, i, j.$$

Consider a symmetric matrix of homogeneous polynomials on $\mathbb{P}^3$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$$a_{ij} = a_{ji}, \quad \deg(a_{ii}) = d_i, \quad \deg(a_{ij}) = \frac{d_i + d_j}{2}. \tag{12}$$

Put

$$d = \sum_i d_i, \quad r_i = \frac{d - d_i}{2}, \quad s_i = \frac{d + d_i}{2}, \tag{13}$$

$$\mathscr{E} = \mathscr{O}(r_1) \oplus \mathscr{O}(r_2) \oplus \mathscr{O}(r_3). \tag{14}$$

Then A determines a symmetric map between graded free bundles

$$A\colon \mathscr{E}(-d) = \mathscr{O}(-s_1) \oplus \mathscr{O}(-s_2) \oplus \mathscr{O}(-s_3) \to \mathscr{E}^\vee = \mathscr{O}(-r_1) \oplus \mathscr{O}(-r_2) \oplus \mathscr{O}(-r_3)$$

hence a line bundle valued map

$$\mathrm{Sym}^2 \mathscr{E} \to \mathscr{O}(d)$$

determining a conic bundle $Y \subset \mathbb{P}(\mathscr{E}) \to \mathbb{P}^3$ if the entries of A do not vanish simultaneously in any point of $\mathbb{P}^3$. Such a conic bundle will be called of *graded-free-type*.

Example 3.2. If $Y \subset \mathbb{P}^5$ is a cubic hypersurface containing a line $\ell \subset \mathbb{P}^5$ the projection $\mathbb{P}^5 \dashrightarrow \mathbb{P}^3$ from ℓ is resolved by the blowup $\widetilde{\mathbb{P}}^5$ of $\mathbb{P}^5$ along ℓ. The resulting morphism $\widetilde{\mathbb{P}}^5 \to \mathbb{P}^3$ has the structure of a projective bundle $\mathbb{P}(\mathscr{E})$, where $\mathscr{E} = \mathscr{O}(1) \oplus \mathscr{O}(2) \oplus \mathscr{O}(2)$. Restricting this morphism to the blowup $\widetilde{Y} \subset \widetilde{\mathbb{P}}^5$ of Y along ℓ, then $\widetilde{Y} \to \mathbb{P}^3$ is a conic bundle of graded-free-type $(3, 1, 1)$, cf. [Tog40]. It does not seem possible to apply Theorem 2.6 to degenerations of conic bundles of this type.

We now derive a result saying that certain discriminant surfaces F of conic bundles of graded-free-type over $\mathbb{P}^3$ split if pulled back via a suitable double cover.

Definition 3.3. A point p on a surface F in $\mathbb{P}^3$ is called a node if

$$\widehat{\mathscr{O}}_{F,p} \simeq k[\![x, y, z]\!]/(xy - z^2).$$

Proposition 3.4. *Let F be a surface in $\mathbb{P}^3$ with at most nodes as singularities. Suppose that for a desingularization $\tilde{F}$ of F, $H^1_{ét}(\tilde{F}, \mathbb{Z}/2) = 0$, or equivalently, $H^1_{\mathrm{nr}}(k(F)/k, \mathbb{Z}/2) = 0$. Let G be a "contact surface" to F, i.e., as schemes $G \cap F = 2C$ for some curve C on F, and suppose, moreover, that G has even multiplicity α_i at every node p_i of F (this also allows $\alpha_i = 0$ of course, whence G does not pass through that particular node). Assume that G has even degree. Then F splits in the double cover of $\mathbb{P}^3$ branched in G.*

Proof. The double cover of $\mathbb{P}^3$ is defined by adjoining a square root of $T := G/X_0^{\deg G}$ to the function field $k(\mathbb{P}^3) = k(X_1/X_0, X_2/X_0, X_3/X_0)$. Let $t \in k(F)$ be the restriction of T to F. We claim that t viewed as an element of

$$H^1(k(F), \mathbb{Z}/2) = k(F)^\times / k(F)^{\times 2}$$

is unramified with respect to every divisorial valuation ν of $k(F)$. Since we assumed that $H^1_{\mathrm{nr}}(k(F), \mathbb{Z}/2) = 0$, this will imply that t is a square, and the cover of F determined by t splits. By Proposition 2.1 we only have to check ν's corresponding to irreducible curves on a smooth model $\pi\colon \tilde{F} \to F$ where we have blown up all nodes p_i to (-2) curves A_i. Then the claim follows since

$$\pi^*(2C) \equiv 2C' + \sum_i \alpha_i A_i,$$

where C' is the strict transform of C on $\tilde{F}$. See also [Cat81, proof of Prop. 2.6]. ☐

Remark 3.5. If G, F meet all the requirements of Proposition 3.4 except that some α_i is not even, say $\alpha_i = 1$ so that G is smooth at p_i, then the cover of F will *not* split since t will vanish to order 1 along A_i in that case. In particular, the intersection curve C cannot locally analytically look like one line of a ruling in a cone at a node p_i if we want the splitting.

Remark 3.6. In the nicest situation, the hypotheses of Proposition 3.4 will be satisfied in such a way that at a node p, C locally analytically looks like two lines of the ruling of a cone.

Example 3.7. We will now analyze the example in [HPT16], which is a divisor Y_{HTP} of bi-degree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$, in light of Proposition 3.4. In [HPT16], the authors used the structure of Y_{HPT} as a quadric surface fibration over $\mathbb{P}^2$, given by the projection onto the first factor. We will use its conic bundle structure over $\mathbb{P}^3$ given by projection onto the second factor. More precisely, Y_{HPT} is defined by

$$YZ\,S^2 + XZ\,T^2 + XY\,U^2 + (X^2+Y^2+Z^2-2(XY+XZ+YZ))\,V^2 = 0, \tag{15}$$

where we denote homogeneous coordinates $(S:T:U:V)$ in $\mathbb{P}^3$ and $(X:Y:Z)$ in $\mathbb{P}^2$.

This conic bundle over $\mathbb{P}^3$ is defined, after rescaling the coordinate $V \mapsto \sqrt{2}V$, by the graded matrix (up to a scalar multiple)

$$\begin{pmatrix} V^2 & U^2-V^2 & T^2-V^2 \\ U^2-V^2 & V^2 & S^2-V^2 \\ T^2-V^2 & S^2-V^2 & V^2 \end{pmatrix}. \tag{16}$$

The discriminant is a sextic surface $D \subset \mathbb{P}^3$ defined by the determinant

$$4V^6 - 4(S^2+T^2+U^2)V^4 + (S^2+T^2+U^2)^2V^2 - 2S^2T^2U^2 = 0 \tag{17}$$

which has two irreducible cubic surfaces as components $D_\pm$, defined by

$$2V^3 - V(S^2+T^2+U^2) \pm \sqrt{2}STU = 0. \tag{18}$$

Each component $D_\pm$ has four nodes and no other singular points, hence up to projective equivalence, is isomorphic to the Cayley nodal cubic surface. In fact, given their equations, the surfaces $D_\pm$ are in the family of tetrahedral Goursat surfaces [G1887], which constitute one of the standard forms for the Cayley nodal cubic. The nodes of the component $D_\pm$ are at the points

$$(1:1:1:\pm\tfrac{1}{\sqrt{2}}), (1:-1:-1:\pm\tfrac{1}{\sqrt{2}}), (-1:1:-1:\pm\tfrac{1}{\sqrt{2}}), (-1:-1:1:\pm\tfrac{1}{\sqrt{2}}). \tag{19}$$

Over each node of the component $D_\pm$, the quadratic form q has rank 1. The only other points where the rank of q drops to 1 are the six points

$$\Sigma := \{(\pm\sqrt{2}:0:0:1), (0:\pm\sqrt{2}:0:1), (0:0:\pm\sqrt{2}:1)\}.$$

Away from these 14 points, q has rank 2 on D.

The components of the discriminant meet in a curve $D_+ \cap D_-$, which is a strict normal crossings curve of degree 9 in $\mathbb{P}^3$, composed of an arrangement of 3 conics and 3 lines as in Figure 1. The equations of the components of $D_+ \cap D_-$ are:

$$\begin{aligned} &\tilde{M}_1\colon (U = S^2+T^2-2V^2 = 0), && \tilde{L}_1\colon (U = V = 0),\\ &\tilde{M}_2\colon (T = S^2+U^2-2V^2 = 0), && \tilde{L}_2\colon (T = V = 0),\\ &\tilde{M}_3\colon (S = T^2+U^2-2V^2 = 0), && \tilde{L}_3\colon (S = V = 0). \end{aligned} \tag{20}$$

Each two of the three conics intersect in two points, and the resulting set of six points coincides with Σ.

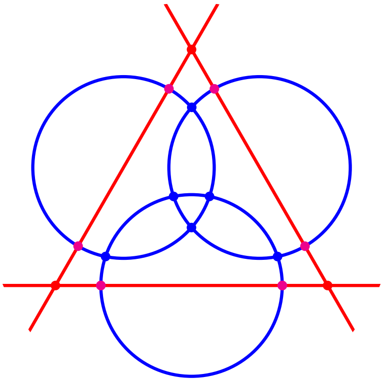

FIGURE 1. The arrangement of components of the intersection of irreducible components $D_+ \cap D_-$ of the discriminant of the conic bundle associated to the example Y_{HPT} in [HPT16].

Although we will verify it more easily in our geometric discussion below, placing this example in the context of Proposition 3.4, the algebraically inclined reader can verify already at this stage that Theorem 2.6 applies to Y_{HPT}, as follows.

By taking successive quotients of increasing minors, we can diagonalize the quadratic form q over $k(\mathbb{P}^3)$ (though still using homogeneous coordinates) as

$$q \sim \langle V^2, (-U^2 + 2U^2V^2)/V^2, D/(-U^2 + 2U^2V^2)\rangle$$

where by abuse of notation, D denotes the homogeneous equation for the discriminant. Hence, we have

$$\alpha = (U^2 - 2V^2, D)$$

in $\operatorname{Br} k(\mathbb{P}^3)$. Hence over the generic point of each component $D_\pm$ of D, we have residue $\gamma_\pm = \partial_{D_\pm}\alpha = (U^2 - 2V^2)$. We know that each residue $\gamma_\pm$ is nontrivial. Indeed, one verifies that $\gamma_\pm$ ramifies along valuations that are centered at the isolated singular points of $D_\pm$, i.e., along the exceptional divisors of a minimal resolution of $D_\pm$.

It is easy, but cumbersome, to check that $\gamma_\pm$ has no further residues along components of $D_+ \cap D_-$ (which follows from the fact that the quadratic form q has rank 2 generically over each component of $D_+ \cap D_-$) and that for each component C of $D_+ \cap D_-$, the residue class is a square in the residue field $k(C)$. Hence, Theorem 2.6 gives that Y_{HPT} has unramified Brauer group $\mathbb{Z}/2\mathbb{Z}$.

We now analyze the conic bundle Y_{HPT} in a more geometric way, establishing the connection to Proposition 3.4.

The first observation is that if we take another copy of $\mathbb{P}^3$ with coordinates X_0, X_1, X_2, X_3 and consider the matrix

$$M = \begin{pmatrix} X_0 & X_1 & X_2 \\ X_1 & X_0 & X_3 \\ X_2 & X_3 & X_0 \end{pmatrix}, \tag{21}$$

then M defines a linear determinantal conic bundle over that $\mathbb{P}^3$ with discriminant $\det M$ a Cayley cubic F with nodes at

$$\nu_0 = (1:1:1:1), \quad \nu_1 = (1:-1:-1:1),$$
$$\nu_2 = (1:1:-1:-1), \quad \nu_3 = (1:-1:1:-1).$$

The conic bundle given by the matrix (16) is the pull-back of this linear determinantal conic bundle via the degree 8 cover

(22)

$$\varphi\colon \mathbb{P}^3_{(S:T:U:V)} \to \mathbb{P}^3_{(X_0:X_1:X_2:X_3)}$$
$$(S:T:U:V) \mapsto (X_0:X_1:X_2:X_3) = (V^2 : U^2 - V^2 : T^2 - V^2 : S^2 - V^2).$$

The branch locus of this cover is given by a tetrahedron of planes in $\mathbb{P}^3$ given by

$$\begin{aligned} G_0 &= \{X_0 = 0\}, \\ G_1 &= \{X_0 + X_1 = 0\}, \\ G_2 &= \{X_0 + X_2 = 0\}, \\ G_3 &= \{X_0 + X_3 = 0\}. \end{aligned} \tag{23}$$

We write $G = \bigcup_i G_i$. Let us give names to six lines on the Cayley cubic F

$$\begin{aligned} M_1 &= \{X_0 + X_1 = 0,\ X_2 + X_3 = 0\}, \\ M_2 &= \{X_0 + X_2 = 0,\ X_1 + X_3 = 0\}, \\ M_3 &= \{X_0 + X_3 = 0,\ X_2 + X_1 = 0\}, \\ L_1 &= \{X_0 = X_1 = 0\}, \\ L_2 &= \{X_0 = X_2 = 0\}, \\ L_3 &= \{X_0 = X_3 = 0\} \end{aligned} \tag{24}$$

and write

$$L = \bigcup_i L_i, \quad M = \bigcup_j M_j.$$

Then L and M are two triangles of lines in F that are "circumscribed around each other", in the sense that L_i meets M_i in a point different from the vertices of M, and L_i does not meet M_j for $i \neq j$. Moreover, the nodes ν_1, ν_2, ν_3 form the vertices of the triangle M. We have the following scheme-theoretic intersections:

$$\begin{aligned} G_0 \cap F &= L, \\ G_i \cap F &= 2M_i + L_i, \quad i = 1,2,3, \\ G \cap F &= 2L + 2M. \end{aligned} \tag{25}$$

So the G_i, $i = 1,2,3$, are tangent to F in M_i, and G itself is singular along L, $G_i \cap G_0 = L_i$, $i = 1,2,3$. Note that the curve $C := L + M$ is Cartier everywhere, even at the nodes. The node $\nu_0 = (1:1:1:1)$ is not in G at all.

In other words, F, G, and C verify all the hypotheses of Proposition 3.4! The eight to one cover φ in (22) factors into a double cover to which Proposition 3.4 applies, and a residual four to one cover. This explains the splitting of the discriminant conceptually for the example Y_{HPT}.

The eight singular points of D_+ and D_- (both Cayley cubics) are the preimages under φ of ν_0. In fact, the cover is étale locally above ν_0. The following formulas hold for the (reduced, set-theoretic) preimages:

$$\begin{aligned} \varphi^{-1}(L_i) &= \tilde{L}_i, \\ \varphi^{-1}(M_i) &= \tilde{M}_i. \end{aligned} \tag{26}$$

We have

$$\varphi^{-1}(\{\nu_1, \nu_2, \nu_3\}) = \Sigma.$$

Let us now verify that the double covers of the curves $\tilde{L}_i$ and $\tilde{M}_j$ induced by the conic bundle given by (16) decompose into two components. Indeed, look at the double covers of the L_i induced by the conic bundle given by (21) first. Then these already split into two components, as is easy to see. For example, taking the line L_1 with homogeneous coordinates X_2, X_3, and fiber coordinates $(a:b:c)$ in the trivial $\mathbb{P}^2$ bundle that the conic bundle given by (16) naturally embeds into, the preimage of L_1 decomposes as

$$c = 0, \quad X_2 a + X_3 b = 0.$$

Similarly for L_2, L_3. So also the double covers of the curves $\tilde{L}_i$ decompose. The double covers of the curves M_j on the contrary are irreducible conics $M_j^\sharp$, the covers $M_j^\sharp \to M_j$ being branched in the two nodes of F lying on M_j. However, if we pull-back the cover $M_j^\sharp \to M_j$ via the cover $\tilde{M}_j \to M_j$, then it becomes reducible (since $\tilde{M}_j$ factors through a double cover square isomorphic to $M_j^\sharp$ over M_j). So all the hypotheses of Corollary 2.9, including the "splitting condition" for the curves arising as irreducible components of some $S_i \cap S_j$, are verified. So we see again that the unramified Brauer group of Y_{HPT} is equal to $\mathbb{Z}/2\mathbb{Z}$.

In [HPT16], the authors show that Y_{HPT} has a Chow universally trivial resolution of singularities, by an explicit computation. The results of Section 6 give a new streamlined proof of this result. Using [Voi15] and [CT-P16], one obtains that the very general divisor of bi-degree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ is not stably rational. On the other hand, some such hypersurfaces, even smooth ones, are shown to be rational in [HPT16].

Remark 3.8. The difficulty in using this approach, or, more precisely, Proposition 3.4, for the construction of new examples to which Theorem 2.6 applies is that the double cover B of $\mathbb{P}^3$ branched in G is usually both nonrational and has nontrivial $H^3_{\text{ét}}(B, \mathbb{Z}/2)$. In cases where B is at least unirational, one can pull back further to a rational B' dominating B, but also this will usually have $H^3_{\text{ét}}(B', \mathbb{Z}/2)$ nontrivial.

4. Reducibility of the discriminant: 2nd method

There is another construction of conic bundles, again using the theory of contact of surfaces, to which Corollary 2.9 potentially applies. The advantage of this method is that it works over the base $B = \mathbb{P}^3$ and that it produces conic bundles of graded-free-types with reducible discriminant surfaces directly, and such that the conics will generically be two distinct lines over the intersections of discriminant components. The subtle condition one must still somehow ensure (e.g., by adjusting the free parameters in the construction) is the splitting condition on the covers of the curves that make up the irreducible components of the intersection of two discriminant surfaces. But this can also be translated entirely into the projective geometry of the configuration, and we will deal with it at the end of this section.

Proposition 4.1. *Consider symmetric matrices over* $\mathbb{P}^3$

$$A = \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} \\ a_{0,1} & a_{1,1} & a_{1,2} \\ a_{0,2} & a_{1,2} & a_{2,2} \end{pmatrix}, \qquad B = \begin{pmatrix} b & c \\ c & d \end{pmatrix},$$

defining symmetric maps between graded-free vector bundles. Let

$$N = \begin{pmatrix} c^2 a_{0,0} - b \det A & c a_{0,1} & c a_{0,2} \\ c a_{0,1} & a_{1,1} & a_{1,2} \\ c a_{0,2} & a_{1,2} & a_{2,2} \end{pmatrix}.$$

If in this situation

$$d = \det \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{1,2} & a_{2,2} \end{pmatrix},$$

then N also gives a symmetric map between graded-free vector bundles and

$$\det N = -(\det A)(\det B).$$

Proof. First notice that

$$\begin{aligned} 2\deg(c) + \deg(a_{0,0}) &= \deg(b) + \deg(d) + \deg(a_{0,0}) \\ &= \deg(b) + \deg(a_{1,1}) + \deg(a_{2,2}) + \deg(a_{0,0}) \\ &= \deg(b) + \deg(\det(A)). \end{aligned}$$

Then evaluate $\det N$ and compare. ☐

Remark 4.2. For the interested reader we sketch how the above construction was found. Even though the concepts are not used in the proof, this construction relies on matrix factorizations and Catanese's theory of contact of surfaces [Cat81]:

The minimal free resolution of a coherent sheaf on a hypersurface $X = \{f = 0\} \subset \mathbb{P}^n$ over the coordinate ring of X becomes periodic after a finite number of steps. If the sheaf is arithmetically Cohen–Macaulay (ACM) with support equal to X, the resolution is periodic. The differentials are given by square matrices P, resp., Q corresponding to maps from F to G, resp., G to F for some graded free modules F and G, with $PQ = f\mathrm{id}_G$ and $QP = f\mathrm{id}_F$. Furthermore the determinants of P and Q vanish on X. The pair (P, Q) with the above properties is called a *matrix factorization* of f [Ei80, Thm. 6.1].

Dolgachev [Dol12, Section 4.2] observes that one obtains symmetric matrices in this way if one starts with an arithmetically Cohen–Macaulay symmetric sheaf. So our problem of finding a symmetric matrix with given reducible determinant X can be reduced to finding an appropriate sheaf on X.

On the other hand, Catanese observed that for a symmetric graded $n \times n$ matrix each diagonal $(n-1) \times (n-1)$ minor defines a contact surface to the determinant of the matrix. Furthermore the square root of the contact curve is defined by the $(n-1) \times (n-1)$ minors of the $(n-1) \times n$ matrix obtained by deleting the line that is not involved in the minor defining the contact surface. In our construction above

$$d = \det \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{1,2} & a_{2,2} \end{pmatrix}$$

is a contact surface to both $\det A$ and $\det B$. The contact curves are defined by the 2×2 minors of

$$\begin{pmatrix} a_{0,1} & a_{1,1} & a_{1,2} \\ a_{0,2} & a_{1,2} & a_{2,2} \end{pmatrix}$$

and the 1×1 minors of

$$\begin{pmatrix} d & c \end{pmatrix}.$$

The ideal sheaves of these curves are ACM (since they are determinantal) and symmetric (since they are contact curves).

Notice now that d is also contact to $(\det A)(\det B)$. Furthermore the contact curve is the union of the two contact curves above. If this union is also ACM we can obtain a symmetric matrix N whose determinant vanishes on $(\det A)(\det B)$ via matrix factorization.

In our case the union of the curves is defined by

$$\begin{pmatrix} ca_{0,1} & a_{1,1} & a_{1,2} \\ ca_{0,2} & a_{1,2} & a_{2,2} \end{pmatrix}.$$

Indeed, if c is nonzero, we obtain the equations of the first curve. If $c = 0$ two of the minors vanish automatically and the third is just d. So we obtain $d = c = 0$ as the second component. This shows that the union of contact curves is again ACM and we obtain the above formula via matrix factorization.

In a certain sense this is a generalization of the construction of Artin and Mumford in [AM72] to $\mathbb{P}^3$.

Note that N defines a conic bundle of graded-free-type if the rank of N is never zero in a point of $\mathbb{P}^3$.

Remark 4.3. Notice that if in the above construction A, B, and N define conic bundles, then the restriction of the conic bundle defined by N to $\det A$ is birationally the same as the one defined by A.

Remark 4.4. In order to apply our Theorem 2.6, or rather Corollary 2.9, to the situation above we must find A and B such that

a) $\det A$ and $\det B$ are irreducible (this is an open condition);
b) $\det A$ and $\det B$ are smooth in the intersection curve $\overline{D} = \{\det A = \det B = 0\}$ (this is an open condition);
c) the double cover of $\det A$ and $\det B$ induced by N is nontrivial (this is also an open condition);
d) N has rank 2 generically on each component of $\overline{D}$;
e) the double cover of the intersection curve $\overline{D}$ induced by N is trivial (this is a closed condition).

The hard part here is the last condition. In the next section we will show how one can satisfy this closed condition via an appropriate construction. The open conditions will then be checked by a computer program for a single example.

5. Triviality of the conic bundle on the intersection curve

The purpose of this section is to construct examples of conic bundles with the properties listed in Remark 4.4 so that we can apply Corollary 2.9. The roadmap for this section is as follows.

- Proposition 5.1 is a sufficient geometric condition to ensure property e) of Remark 4.4 will be satisfied.
- The construction proceeds by taking one of the irreducible components of the discriminant to be a Cayley cubic surface. The results from Proposition 5.2 to Example 5.8 are classical facts about the Cayley cubic surface needed in what follows.
- Using a rational parametrization of the Cayley cubic surface, we subsequently construct, on $\mathbb{P}^2$, a candidate for the intersection curve of the Cayley cubic with the sought-after second discriminant component that has the right determinantal format for Proposition 4.1 to apply, and such that the components of the intersection curve satisfy the condition of Proposition 5.1.
- Proposition 5.9 to Remark 5.13 are conditions for curves in $\mathbb{P}^2$ to have a determinantal representation that makes them candidates for intersection curves of discriminant components. These results also give a method to construct such determinantal representations.
- The rest of the section is then concerned with the construction of our example.

Proposition 5.1. *In the notation of Proposition* 4.1 *let*

$$\overline{D} = \{\det A = \det B = 0\} \subset \mathbb{P}^3$$

be the intersection curve of the two discriminant components. If all components of $\overline{D}$ *are rational and do not intersect the rank* 1 *locus of* A*, and, moreover,* N *has rank* 2 *generically on each component of* $\overline{D}$*, then the double cover of each component of* $\overline{D}$ *induced by* N *is trivial.*

Proof. By Remark 4.3 the double cover of $\overline{D}$ induced by N is birationally the same as the one induced by A. Since $\overline{D}$ does not intersect the rank 1 locus of A this double cover is étale. Since there are no nontrivial étale double covers of $\mathbb{P}^1$ and $\overline{D}$ consists of rational components, the double cover induced by A, and with it the one induced by N, is trivial. $\square$

For the remainder of this section we restrict to the case where all $a_{i,j}$ are linear and $\det A$ is the Cayley cubic. We can change coordinates so that the Cayley cubic is in the form (21), and equivalently, find an invertible matrix S

such that

$$SAS^t = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_0 & x_3 \\ x_2 & x_3 & x_0 \end{pmatrix}.$$

For our construction we will use the fact that the Cayley cubic is rational.

Proposition 5.2. *Let $L_1, \dots, L_4$ be 4 general linear forms defining 4 general lines in $\mathbb{P}^2$ intersecting in 6 distinct points. Consider the cubic polynomials*

$$Y_i = \prod_{i \neq j} L_j$$

and

$$\begin{aligned} X_0 &= -Y_0 + Y_1 + Y_2 + Y_3, \\ X_1 &= -Y_0 - Y_1 - Y_2 + Y_3, \\ X_2 &= Y_0 - Y_1 + Y_2 + Y_3, \\ X_3 &= Y_0 + Y_1 - Y_2 + Y_3. \end{aligned}$$

Then the image of $\mathbb{P}^2$ under the rational map $\varphi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ defined by the linear system $|\langle X_0, X_1, X_2, X_3 \rangle|$ is the Cayley cubic.

Proof. Setting $x_i = X_i$ in SAS^t, the evaluation of the determinant gives zero. □

Remark 5.3. Recall the following facts from classical algebraic geometry:

a) The Cayley cubic has 4 nodes. They form the rank 1 locus of A.
b) The four lines $L_1, \dots, L_4$ are contracted by φ. Their images are the 4 nodes.
c) The 6 base points are blown up and their images are 6 lines in $\mathbb{P}^3$. These 6 lines form a tetrahedron with the 4 nodes as vertices.

Notation 5.4. Let $\sigma \colon \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ in the 6 base points above. With this we have the following diagram:

$$\begin{tikzcd} & \widetilde{\mathbb{P}}^2 \arrow[dl, "\sigma"'] \arrow[dr, "\pi"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\varphi"] & & X_3 \subset \mathbb{P}^3 \end{tikzcd}$$

where $X_3 \subset \mathbb{P}^3$ denotes the Cayley cubic. If $C \subset \mathbb{P}^2$ is a plane curve, we denote by $\widetilde{C} \subset \widetilde{\mathbb{P}}^2$ its strict transform and by

$$\overline{C} := \pi(\widetilde{C}) \subset X_3 \subset \mathbb{P}^3$$

the image of $\widetilde{C}$ in $\mathbb{P}^3$. Furthermore, denote by $E_{i,j} \subset \widetilde{\mathbb{P}}^2$ the exceptional divisor over the intersection point of L_i and L_j, and by H the class of the pull back of a line in $\mathbb{P}^2$ to $\widetilde{\mathbb{P}}^2$.

We are interested in curves on the Cayley cubic that do not intersect the nodes.

Lemma 5.5. *Let $\widetilde{C} \subset \widetilde{\mathbb{P}}^2$ be the strict transform of a curve C in $\mathbb{P}^2$ not containing any of the L_i as components, and suppose its class is*

$$\widetilde{C} \equiv \alpha H - \sum_{i<j} \beta_{i,j} E_{i,j}.$$

Then the image $\overline{C} = \pi(\widetilde{C}) \subset \mathbb{P}^3$ avoids the nodes of the Cayley cubic if and only if $\beta_{i,j} = \beta_{k,l}$ for all indices with $\{i,j,k,l\} = \{1,2,3,4\}$ and $\alpha = \sum_j \beta_{i,j}$ for every i.

Proof. Since the preimage of the nodes are the lines L_i we want $\widetilde{C}.\widetilde{L_i} = 0$ for all i where $\widetilde{L_i}$ is the strict transform of L_i on the blowup. This gives the following linear system of equations:

$$\begin{pmatrix} 1 & -1 & 0 & -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 & -1 & 0 \end{pmatrix} \cdot (\alpha, \beta_{1,2}, \beta_{3,4}, \beta_{1,3}, \beta_{2,4}, \beta_{1,4}, \beta_{2,3})^t = 0.$$

The solution of this system is the one claimed above. ☐

Definition 5.6. We call a curve $C \subset \mathbb{P}^2$ of type (b_1, b_2, b_3) if its strict transform has class

$$\widetilde{C} \equiv (b_1 + b_2 + b_3)H - b_1(E_{1,4} + E_{2,3}) - b_2(E_{2,4} + E_{1,3}) - b_3(E_{3,4} + E_{1,2}).$$

If C does not contain any of the lines L_i as components, then the image $\overline{C} \subset \mathbb{P}^3$ of such a curve avoids the nodes of the Cayley cubic by Lemma 5.5.

We collect some numerical facts about these curves.

Lemma 5.7. *Let $C \subset \mathbb{P}^2$ be a curve of type (b_1, b_2, b_3) and let $\widetilde{C}$ be its strict transform and $\overline{C} \subset \mathbb{P}^3$ its image. Then*

a) *The degree of $\overline{C}$ is $\deg(\overline{C}) = b_1 + b_2 + b_3$.*
b) *The arithmetic genus of $\overline{C}$ is $g_a = \binom{b_1+b_2+b_3}{2} - (b_1^2 + b_2^2 + b_3^2) + 1$.*
c) *The expected number of moduli of C is $\deg(\overline{C}) + g_a$.*

Proof. For the first two items we work on $\widetilde{\mathbb{P}}^2$. The linear system of φ has class $-K = 3H - \sum E_{i,j}$ there; i.e., it consists of curves of type $(1,1,1)$. This is also the anticanonical system. We have

$$\deg \overline{C} = -K.\widetilde{C} = 3(b_1 + b_2 + b_3) - 2b_1 - 2b_2 - 2b_3 = b_1 + b_2 + b_3.$$

The arithmetic genus of $\overline{C}$ is given by the adjunction formula

$$2g_a - 2 = K.\widetilde{C} + \widetilde{C}^2 = -b_1 - b_2 - b_3 + (b_1 + b_2 + b_3)^2 - 2b_1^2 - 2b_2^2 - 2b_3^2.$$

For the number of moduli, we work with plane curves. The dimension of the space of degree $b_1 + b_2 + b_3$ curves in $\mathbb{P}^2$ is $\binom{b_1+b_2+b_3+2}{2}$, the number of conditions for a b_i fold point is $\binom{b_i+1}{2}$. Therefore the expected number of moduli is

$$\binom{b_1 + b_2 + b_3 + 2}{2} - \sum_{i=1}^{3} \binom{b_i + 1}{2},$$

which simplifies to the formula above. ☐

Example 5.8. We have for examples:

type	image in $\mathbb{P}^3$
$(1, 0, 0)$	a line
$(1, 1, 0)$	a plane conic
$(1, 1, 1)$	a plane cubic
$(2, 1, 1)$	an elliptic normal curve of degree 4
$(2, 2, 2)$	a canonical curve, i.e., degree 6 and genus 4
$(1, 2, 3)$	a sextic curve of genus 2

Let us now look at a contact quadric to the Cayley surface.

Proposition 5.9. *Let $Q \subset \mathbb{P}^3$ be a contact quadric defined by a generalized 2×2 diagonal minor of A. Then there exists a line $L_c \subset \mathbb{P}^2$ such that the transform $\sigma_* \pi^*(Q \cap X_3)$ of Q on $\mathbb{P}^2$ is*

$$q = L_c^2 + L_1 + L_2 + L_3 + L_4.$$

Proof. The contact quadric passes through all nodes of X_3 (it is one of the minors defining the ideal of the nodes), so its transform contains the lines $L_1, \ldots, L_4$. Outside of the nodes the contact quadric intersects the Cayley cubic with multiplicity 2. It follows that the transform has the form

$$L_c^2 + L_1 + \cdots + L_4.$$

Since the transform of any quadric is of degree 6 it follows that L_c must be a line. ☐

Notice that the transform of $\{\det B = 0\}$ on $\mathbb{P}^2$ is just the transform of the intersection curve $\overline{D}$ on $\mathbb{P}^2$. To keep with our convention, we denote this by D. In other words, on $\mathbb{P}^2$, we have that D is the determinant of the matrix obtained by forming the transforms of all the entries in B. In view of Proposition 5.1, we would like D to be a union of rational curves. The idea of the construction is now to start with such a D and then try to write it as a determinant. Again we would like to mimic the construction of Artin and Mumford. For this we need a slight generalization of their method to the case where the contact curve is not reduced. For this we need the following technical lemma.

Lemma 5.10. *Let D be a curve of type (d,d,d) with $d \geq 4$ even and $\frac{3d}{2}$ ordinary nodes on L_c. Let f be a generator of the ideal of D and let s be a generator of the ideal of L_c. Suppose that L_c does not pass through any of the base points and that D avoids the intersection points of L_c with the exceptional lines. Let $Z \subset \mathbb{P}^2$ be the subscheme consisting of all the base points with multiplicity $\frac{d}{2}-2$. Assume that the natural map*

$$H^0\left(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}\left(\frac{3d}{2}-6\right)\right) \to H^0\left(\mathbb{P}^2, \mathscr{O}_Z\right) \tag{27}$$

is surjective.

Then there exists a polynomial g on $\mathbb{P}^2$ such that

a) $f \equiv g^2 \mod s^2$,

b) *the curve $\sqrt{D}$ defined by $\{g=0\}$ is of type $(\frac{d}{2}, \frac{d}{2}, \frac{d}{2})$.*

Proof. Choose homogeneous coordinates u, v, s in $\mathbb{P}^2$. Since D has only ordinary nodes on $L_c = \{s=0\}$, hence, in particular, intersects L_c in a divisor that is divisible by 2, there exists a polynomial $g_0 \in k[u,v,s]$ with

$$g_0^2 \equiv f \mod s.$$

More precisely, we choose g_0 such that it vanishes at the nodes of D on L_c and has multiplicity $\frac{d}{2}$ in all base points. This is clearly possible for $d \geq 4$ since an ordinary multiple point of order e imposes $e(e+1)/2$ conditions on plane curves, and g_0 has degree $3d/2$. We therefore have a polynomial $f_1 \in K[u,v,s]$ such that

$$f - g_0^2 = f_1 s.$$

Taking the derivative with respect to s we get

$$\frac{df}{ds} - 2g_0\frac{dg_0}{ds} = f_1 + \frac{df_1}{ds}s.$$

For every point $P \in L_c \cap D$ all derivatives of f vanish (since D has a node there). Also g_0 vanishes at all such points by construction. Therefore the equation above also gives $f_1(P)=0$. This implies that g_0 divides f_1 modulo s; i.e., there exists a g_1 such that

$$2g_0g_1 \equiv f_1 \mod s.$$

We obtain

$$(g_0+g_1s)^2 \equiv g_0^2 + 2g_0g_1s \equiv g_0^2 + f_1s \equiv f \mod s^2.$$

We now want to find a $g_2 \in K[u,v,s]$ such that

$$g = g_0 + g_1s + g_2s^2$$

defines a curve of type $(\frac{d}{2}, \frac{d}{2}, \frac{d}{2})$. Notice that this leads to an affine linear system of equations for the coefficients of g_2. To prove the solvability of this system we have to analyze the geometric situation in more detail.

First notice that $\{f_1 = 0\}$ is a curve of degree $3d - 1$ that passes with multiplicity d through each base point (since L_c does not contain any of the base points). Now there are 3 base points on each exceptional line. It follows by Bezout's theorem that $\{f_1 = 0\}$ contains all 4 exceptional lines as components. We can therefore write

$$f_1 = f_1' l_1 l_2 l_3 l_4$$

where l_i is an equation for L_i. Furthermore, since none of the exceptional lines pass through any of the nodes of D, we have that g_0 divides not only f_1, but also f_1' modulo s. It follows that there is a polynomial g_1' with

$$2g_0 g_1' \equiv f_1' \mod s$$

and

$$g_1 = g_1' l_1 l_2 l_3 l_4.$$

We have $\deg g_1 = \deg f_1 - \deg g_0 = 3d - 1 - \frac{3d}{2} = \frac{3d}{2} - 1$ and therefore

$$\deg g_1' = \frac{3d}{2} - 5.$$

Now, the surjectivity of the map (27) implies the existence of a g_2' of degree $\frac{3d}{2} - 6$ such that

$$g_1' + s g_2'$$

has multiplicity $\frac{d}{2} - 2$ in each base point. With $g_2 := g_2' l_1 l_2 l_3 l_4$ we obtain that

$$\{g_1 + s g_2 = 0\}$$

passes through all base points with multiplicity $\frac{d}{2}$. Since the same is true for g_0 we get that

$$g = g_0 + s g_1 + s^2 g_2$$

defines a curve of type $(\frac{d}{2}, \frac{d}{2}, \frac{d}{2})$. □

With this, we get an instance of our generalized version of the Artin–Mumford method.

Proposition 5.11. *Let D be a curve of type (d, d, d) with $d \geq 4$ even. Assume that D has $\frac{3d}{2}$ ordinary nodes on L_c, L_c contains none of the base points, D avoids the intersection points of L_c with the exceptional lines, and that the map (27) is surjective. Then there exists a matrix*

$$B = \begin{pmatrix} q & r \\ r & t \end{pmatrix}$$

with $\{q = 0\}$ *the transform of the contact quadric* Q*,* $\{r = 0\}$ *defining* $\sqrt{D}$*, and* $\{t = 0\}$ *of type* $(d-2, d-2, d-2)$*, such that* D *is defined by* $\det B$*.*

Proof. Let f be a defining equation of D. By Lemma 5.10 there exists a curve $\sqrt{D}$ with defining equation $r = 0$ such that $f \equiv r^2 \mod s^2$. Therefore $f - r^2$ is divisible by s^2. Now $f - r^2$ vanishes on each line L_i with multiplicity d in the three base points that lie on L_i. Furthermore $f - r^2$ vanishes with multiplicity 2 on the intersection $L_c \cap L_i$. So $f - r^2$ vanishes with multiplicity at least $3d + 2$ on L_i. Bezout's theorem implies then that $f - r^2$ vanishes also on L_i. In total $f - r^2$ vanishes on $\{q = 0\} = L_c^2 + L_1 + \cdots + L_4$ and is therefore divisible by q. Set

$$t := -\frac{f - r^2}{q}.$$

With this we get

$$-f = qt - r^2 = \det \begin{pmatrix} q & r \\ r & t \end{pmatrix}.$$

□

Lemma 5.12. *The map* (27) *is surjective for* $d = 6$*.*

Proof. For $d = 6$ the scheme Z is the union of all base points $P_1, \dots, P_6$ with multiplicity 1 and the map (27) is

$$H^0\left(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(3)\right) \to H^0\left(\mathbb{P}^2, \mathscr{O}_Z\right).$$

For the surjectivity of this map we construct cubics C_i that pass through all P_j with $j \neq i$ but not through P_i.

For this, notice that there is no quadric that passes through all six P_i. Indeed, assuming the contrary we would get a quadric Q that passes through 3 points on every exceptional line and must therefore contain all 4 such lines as a factor, which is a contradiction.

For each $i \in \{1, \dots, 6\}$ there exists a quadric $Q_i \neq 0$ passing through the five P_j with $j \neq i$. Since there is no Q through all six base points, we have $Q_i(P_i) \neq 0$. Now choose a line that does not pass through P_i and we get cubics $C_i = L_i Q_i$ with the desired properties. □

The next problem in our construction is to find curves D of type (d, d, d) with all components rational.

Remark 5.13. The existence of such curves D of type (d, d, d), with components rational, and with $\frac{3d}{2}$ nodes on L_c is expected. Indeed, the arithmetic genus g_a of the image of $\overline{D}$ in $\mathbb{P}^3$ is

$$g_a = \binom{3d}{2} - 3d^2 + 1 = \frac{3d(3d-1)}{2} - 3d^2 + 1 = \frac{3}{2}d^2 - \frac{3}{2}d + 1 = 3\binom{d}{2} + 1,$$

in particular $g_a > \frac{3d}{2}$. For D to be rational we need it to have g_a nodes. This poses g_a conditions. Furthermore, $\frac{3d}{2}$ of them should lie on L_c. This poses a further $\frac{3d}{2}$ condition. So we have $\frac{3d}{2} + g_a$ conditions and $3d + g_a$ moduli. So we expect such curves to exist.

Unfortunately, this is not enough to apply Theorem 2.6. For this we must also show that a number of open conditions are satisfied. We propose to do this by constructing a concrete example over a finite field $\mathbb{F}_p$ along the lines suggested so far in this section and then check the open conditions for this example.

Now, finding a rational curve as described above explicitly is hard, since the conditions above are highly nonlinear. For example, having a node *somewhere* means that a certain discriminant of high degree in the coefficients of D vanishes. This is a highly nonlinear codimension 1 condition. Having a node *at a given point* on the other hand is a linear codimension 3 condition. So one might try to construct such a curve by prescribing g_a nodes *at given points* (some of them on L_c). Unfortunately this poses

$$3g_a > g_a + 3d$$

conditions, which is larger than the number of moduli.

So we must choose our curves more carefully, which takes up the remainder of this section.

Construction 5.14. Consider the case $d = 6$ with reducible $D = D_1 + D_2 + D_3$ and D_i of type $(1,2,3), (2,3,1)$, and $(3,1,2)$, respectively.

a) Choose points $P_1, \dots, P_6$ and Q_1 on L_c.
b) Choose a curve D_1 of type $(1,2,3)$ with nodes at P_1 and P_2 and vanishing at Q_1. This is possible since the number of projective moduli of such curves is $d + g_a - 1 = 6 + 2 - 1 = 7$ and the number of conditions imposed is $3 + 3 + 1 = 7$. So generically there is only one such curve.
c) D_1 has degree 6 and of the 6 intersection points with L_c we have prescribed 5 so far. Let Q_2 be the remaining intersection point.
d) Choose a curve D_2 of type $(2,3,1)$ with nodes at P_3 and P_4 also passing through Q_1. Again there is generically one such curve.
e) Let Q_3 be the remaining intersection point of D_2 with L_c.
f) Choose a curve D_3 of type $(3,1,2)$ with nodes P_5 and P_6 and passing through Q_2.
g) Let Q_4 be the remaining intersection point of D_3 with L_c.

We can summarize the construction so far in the following table:

	P_1	P_2	P_3	P_4	P_5	P_6	Q_1	Q_2	Q_3	Q_4
D_1	2	2					1	1		
D_2			2	2			1		1	
D_3					2	2		1		1
D	2	2	2	2	2	2	2	2	1	1

Now if $Q_3 = Q_4$ this gives a curve D with 9 nodes on L_c. This is at most a codimension 1 condition. Furthermore for each $i \in \{1, 2, 3\}$ the curve D_i is of arithmetic genus $g_a = 2$ and therefore of geometric genus zero.

Remark 5.15. For reasons not clear to us, the condition $Q_3 = Q_4$ was automatically satisfied in all examples we tried.

Proposition 5.16. *There exists a conic bundle $Y \to \mathbb{P}^3$, defined over a finite field $k_0 = \mathbb{F}_p$, $p = 10007$, defined by a homogeneous 3×3 matrix with entries of degrees*

$$\begin{pmatrix} 7 & 4 & 4 \\ 4 & 1 & 1 \\ 4 & 1 & 1 \end{pmatrix}$$

such that Corollary 2.9 predicts a nontrivial unramified Brauer class for the base change of Y to the closure k of k_0; hence Y is not stably rational (over k).

Proof. Construct a curve $D = D_1 + D_2 + D_3$ as in Construction 5.14 over the finite field k_0 using a computer algebra program. Denote by $\overline{D} = \overline{D}_1 + \overline{D}_2 + \overline{D}_3$ the image of the strict transformation of the previous curves in $\mathbb{P}^3$.

Calculate a matrix representation $\det B$ for D using Proposition 5.11. Find a preimage $\overline{B}$ of B in $\mathbb{P}^3$. The determinant of $\overline{B}$ defines a sextic hypersurface $X_6 \subset \mathbb{P}^3$. Use Proposition 4.1 to construct a matrix N with the degrees claimed. Then check the following:

a) X_6 is irreducible. We do this by checking that the singular locus is finite.
b) X_6 is smooth along $\overline{D}$.
c) The Cayley cubic is smooth along $\overline{D}$.
d) The rank 1 locus of N is finite.
e) The rank 0 locus of N is empty.
f) The curves $\overline{D}_i$ are indeed irreducible and rational. (Our calculation of the geometric genus above relied on the assumption of D being irreducible or at least connected.) We do this by explicitly calculating a parametrization $\mathbb{P}^1 \to \overline{D}_i$.
g) The double cover induced by N is nontrivial on the Cayley cubic and X_6. We do this using Lemma 5.17.

This shows that we can apply Theorem 2.6 in this situation.

A Macaulay2 program for performing the above calculations can be found at [ABBP16]. □

Lemma 5.17. *Let* $\pi\colon Y \to B$ *be a conic bundle defined over* $k_0 = \mathbb{F}_p$*. Let* S *be an irreducible surface in* B*, defined over* k_0*, over which the fibers of* Y *generically consists of two distinct lines. Let* $\tilde{S} \to S$ *be the natural double cover of* S *induced by* π*. Then* $\tilde{S}$ *is irreducible if the following hold: there exist two* k_0*-rational points* $p_1, p_2 \in S$ *such that the fiber of* Y *over* p_1 *splits into two lines defined over* k_0 *whereas the fiber over* p_2 *is irreducible over* k_0 (*and splits in a quadratic extension of* k_0 *only*).

Proof. Under the assumptions the double cover $\tilde{S} \to S$ is defined over k_0. Suppose, by contradiction, that $\tilde{S}$ were (geometrically) reducible. Then the Frobenius morphism F would either fix each irreducible component of $\tilde{S}$ as a set, or interchange the two irreducible components. But since S is defined over k_0, this would mean that F either fixes each of the two lines as a set in every fiber over a k_0-rational point of the base, or F interchanges the two lines in every fiber over a k_0-rational point. This contradicts the existence of p_1, p_2. □

6. Desingularization of conic bundle fourfolds

The conic bundles considered above are singular. In this section, we prove a criterion for the existence of a universally CH_0-trivial desingularization for such conic bundles. Let k be an algebraically closed field of characteristic not 2. First recall the following notion from [A-CT-P] and [CT-P16].

Definition 6.1. A projective variety X over a field k has universally trivial CH_0 if for any extension $L \supset k$, the degree homomorphism $\deg\colon \mathrm{CH}_0(X_L) \to \mathbb{Z}$ is an isomorphism. A morphism $f\colon \tilde{Y} \to Y$ of projective varieties over k is called universally CH_0-trivial if for any overfield $L \supset k$, the pushforward $f_*\colon \mathrm{CH}_0(\tilde{Y}_L) \to \mathrm{CH}_0(Y_L)$ is an isomorphism.

We will make use of the following criterion to check that a resolution of singularities is universally CH_0-trivial.

Proposition 6.2. *A projective morphism* $f\colon \tilde{Y} \to Y$ *of projective varieties over* k *is universally* CH_0*-trivial if for any scheme-theoretic point* ξ *of* Y*, the fiber* $\tilde{Y}_\xi$*, as a scheme over the residue field* $\kappa(\xi)$*, is a projective variety over* $\kappa(\xi)$ *with universally trivial* CH_0*.*

This is [CT-P16, Prop. 1.8]. Moreover, we use this in combination with the following result; cf. [HPT16, Ex. 2].

Proposition 6.3. *A projective, possibly reducible, geometrically connected variety* $X = \bigcup X_i$ *over a field* k *has universally trivial* CH_0 *if each* X_i *is geometrically irreducible,* k*-rational with isolated singularities, and each intersection* $X_i \cap X_j$ *is either empty or has a zero cycle of degree* 1*.*

Now we are ready to state our main result about the existence of universally CH_0-desingularizations of conic bundle fourfolds. If $Y \to B$ is a conic bundle, we colloquially say that Y has a given rank over a point of B to mean that the fibral conic has that rank at the respective point.

Theorem 6.4. *Let* $Y \to \mathbb{P}^3$ *be a conic bundle with reducible discriminant* $X = X' \cup X''$*. Let* $D = X' \cap X''$ *be the intersection curve. Assume:*

- X' *and* X'' *are smooth along* D*.*
- X' *and* X'' *have only isolated nodes as singularities.*
- *The rank of* Y *at all nodes of* X' *and* X'' *is* 1*.*
- $D = D_1 \cup \cdots \cup D_n$ *with* D_i *irreducible reduced.*
- D *has only nodes as singularities.*
- *The rank of* Y *along* D *is* 2 *outside of the nodes of* D*.*
- *The rank of* Y *is* 1 *on each node of the irreducible components* D_i *of* D (*but not necessarily on the intersection points between two irreducible components* D_i *and* D_j *of* D)*.*

Then Y *has a universally* CH_0*-trivial desingularization.*

Remark 6.5. Notice that both the Hassett–Pirutka–Tschinkel example from [HPT16] (see Example 3.7) and our new example (see Proposition 5.16) satisfy these conditions. See [ABBP16] for computational details concerning our new example.

Theorem 6.6. *A very general conic bundle* $Y \to \mathbb{P}^3$ *over* $\mathbb{C}$*, defined by a homogeneous* 3×3 *matrix with entries of degrees*

$$\begin{pmatrix} 7 & 4 & 4 \\ 4 & 1 & 1 \\ 4 & 1 & 1 \end{pmatrix}$$

is not stably rational.

Proof. Follows from Proposition 5.16, Theorem 6.4, Remark 6.5, and the specialization principle in unequal characteristic [CT-P16, Thm. 1.12], as employed in the proof of [CT-P16, Thm. 1.20]. □

To prove the above theorem, some local computations are unavoidable.

Proposition 6.7. *Let* $Y \to \mathbb{P}^3$ *be a conic bundle with reducible discriminant* $X = X' \cup X''$*. Let* $D = X' \cap X''$ *be the intersection curve and let* X' *and* X'' *be smooth along* D*. Let* D *be reduced. Assume furthermore that the conic bundle has rank* 2 *over the smooth locus of* D*. Finally let* $P \in D$ *be a point. Then we have the following local analytic normal forms:*

Geometry of D at P	*Rank of Y at P*	*Normal form*
smooth	2	$x^2 + sty^2 - z^2 = 0$
node	2	$x^2 + sqy^2 - z^2 = 0$
node	1	$x^2 + 2syz + (ty + uz)^2 = 0$

Here $q = s + tu$ is quadratic in the completion $A = k[\![s, t, u]\!]$ of the local ring at P and $(x : y : z)$ are homogeneous coordinates for $\mathbb{P}^2_A$.

Proof. Let M be a 3×3 matrix over A representing Y locally analytically around P.

First assume that Y has rank 2 at P. Then M has rank 2 at P and we can, after a coordinate change on $\mathbb{P}^2_A$, assume that

$$M_P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Therefore, the first 2 diagonal entries are units in A and we can, after a further coordinate change, assume that

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with d in A a local equation for the discriminant of Y.

$\boxed{\text{Case 1.}}$ In the first case of the proposition, D is smooth at P and therefore X' and X'' intersect transversally around P. Consequently, we can change coordinates in A to obtain $X' = \{s = 0\}$ and $X'' = \{t = 0\}$ with s, t linear forms, i.e., $d = st$. This gives the first normal form.

$\boxed{\text{Case 2.}}$ In the second case, D has a node at P and therefore X' and X'' are tangent at P. Let $X' = \{s = 0\}$ and $X'' = \{q = 0\}$. Since X' is smooth at P, we can assume s to be linear. Since $D = \{s = q = 0\}$ has a node in P, it has two smooth normal crossing branches there. We choose t and u to be local linear equations of these branches on $\{s = 0\}$. Then

$$q = tu \mod s,$$

and we can write

$$q = \alpha s + tu.$$

Now since X'' is smooth at P, we see that α must be a unit. Absorbing α into s we obtain $d = s(s + tu)$, which gives the second normal form.

$\boxed{\text{Case 3.}}$ In the third case, Y has rank 1 at P. By evaluating M at P and changing coordinates on $\mathbb{P}^2_A$ as above we can assume

$$M = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}$$

with N a symmetric 2×2 matrix with entries in the maximal ideal of A.

Since D has a node at P we can, as before, assume that the discriminant $\det N = -sq$ with $q = s + 2tu$ and s, t, u linear as above. (The minus sign and the 2 will be convenient later on.)

Now M has rank 2 on $\{s = 0\}$ outside the origin, and rank 1 in the origin.

In other words, N is a matrix, defined locally around the origin in the (t, u)-plane, and has rank 1 everywhere in that plane except at the origin, where it has rank 0 (i.e., vanishes). Let

$$N = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \tag{28}$$

so that $\alpha(t,u)y^2 + 2\beta(t,u)yz + \gamma(t,u)z^2$ is the associated quadratic form. Hence we must have

$$\alpha\gamma - \beta^2 \equiv 0 \tag{29}$$

identically. Now consider the prime factorizations of α, β, γ: if some prime π divides α to odd order, it must divide γ to odd order, too, since it divides the square β^2 to even order. Hence, in that case, π divides all three of them, which contradicts our assumption that the rank of N does not drop to 0 on an entire curve germ through the origin in the (t, u)-plane. Hence, α, γ are coprime squares, and we can write

$$(y\ z)N(y\ z)^t \equiv (t'y + u'z)^2 \mod s \iff N \equiv \begin{pmatrix} t'^2 & t'u' \\ t'u' & u'^2 \end{pmatrix} \mod s$$

with t', u' at least of degree 1, since both vanish at P, and coprime. It follows that we can write N as

$$N = \begin{pmatrix} sf & sg \\ sg & sh \end{pmatrix} + \begin{pmatrix} t'^2 & t'u' \\ t'u' & u'^2 \end{pmatrix}.$$

Then the discriminant of Y is

$$\det M = \det N = s\big(s(fh - g^2) + (ht'^2 + 2gt'u' + fu'^2)\big).$$

Since this is equal to $-s^2 - 2stu$, and t', u' are power series of degree at least 1 in u, t, comparing coefficients yields $fh - g^2 = -1$. We can therefore, after changing the fiber coordinates y and z, assume that

$$\begin{pmatrix} f & g \\ g & h \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The same coordinate change applied to $(t'y + u'z)$ gives $(t''y + u''z)$. We obtain

$$\det M = s\big(-s + 2t''u''\big).$$

Comparing coefficients with $\det M = -sq$ above we see that we can take $\alpha = -1$, $t'' = t$, and $u'' = u$. Then

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^2 & s+tu \\ 0 & s+tu & u^2 \end{pmatrix}, \quad \det M = -s(s+2tu),$$

and we get the claimed normal form. ☐

Now we desingularize in these local coordinates.

Proposition 6.8. *Let $Y \to \mathbb{P}^3$ be a conic bundle with reducible discriminant $X = X' \cup X''$. Let $D = X' \cap X''$ be the intersection curve and let X' and X'' be smooth along D. Assume furthermore that the conic bundle has rank 2 over the smooth locus of D. Finally let $P \in D$ be a point. With the normal forms from Proposition 6.7 we have*

Geometry of D at P	*rank of Y at P*	*Singular Locus*	*Desingularization*
smooth	2	*a line*	*blow up line*
node	2	*2 intersecting lines*	*blow up lines in arbitrary order (but not at the same time)*
node	1	*2 disjoint lines*	*blow up lines in arbitrary order or at the same time.*

In all three cases we have the following geometry. Consider the points $P \in D$ where Y has rank 2. The fiber Y_P over P consists of two lines which intersect in a point $P' \in Y_P$. Let $D' \subset Y$ be the closure of the locus of all such intersection points P'. Then D' is the singular locus of Y. Furthermore the covering $D' \to D$ is $1:1$ over smooth points of D and $2:1$ over rank 1 nodes of D. Over rank 2 nodes of D, D' also has a node.

Proof. These are all straightforward calculations. See [ABBP16] for a Macaulay2 script to perform them. ☐

Remark 6.9. Blowing up the intersection point of the two lines in the case of a rank 2 node does not improve things. While the strict transforms of the two singular lines are separated we obtain a new singular line in the exceptional divisor passing through both of the strict transforms.

Lemma 6.10. *Let $\pi\colon Y \to \mathbb{P}^3$ be a conic bundle with discriminant X a surface having a node at $P \in X$. Assume Y has rank 1 at P and has rank 2 on $X \setminus \{P\}$ locally around P. Then Y is smooth over P and has a local analytic normal form*

$$x^2 + sy^2 + 2tyz + uz^2 = 0$$

where $(x : y : z)$ are homogeneous coordinates on $\mathbb{P}^2_A$ with $A = k[\![s,t,u]\!]$.

Proof. Let M be a 3×3 matrix over A representing Y locally analytically around P. By evaluating M at P and changing coordinates on $\mathbb{P}^2_A$ as above we can assume

$$M = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}$$

with N a symmetric 2×2 matrix with entries in the maximal ideal of A. Let

$$N = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

The lemma follows if we can show that $a = b = c = 0$ defines P as a reduced point because then we can choose a, b, c as local coordinates. Since P is assumed to be a node $\det(N) = 0$, we have that the Jacobian ideal J of $\det(N)$ defines P as a reduced point. Since $J \subset (a, b, c)$ by the product rule for derivatives, our claim follows. The fact that the total space of Y is smooth above P is then a direct calculation. $\square$

Proof of Theorem 6.4. We have to verify the hypotheses of Propositions 6.2 and 6.3 for the resolutions $\tilde{Y} \to Y$ that we produced in Proposition 6.8.

Since the singular locus of X' and X'' consists only of isolated nodes at rank 1 points outside of D, the conic bundle Y is smooth outside of the preimage of D by Lemma 6.10.

Let D' be the closure of the locus of intersection points of lines in fibers over D. By our assumptions in Theorem 6.4 the conditions of Propositions 6.7 and 6.8 are satisfied. Furthermore, the local normal forms studied in these propositions are the only ones that occur. It follows that the singular locus of Y is D'. Let $D' = D'_1 + \cdots + D'_n$ be its decomposition into irreducible components. By Proposition 6.8 these components are birational to the components of D.

We want to blow up the D'_i in arbitrary order to obtain a desingularization. According to Proposition 6.8, the only problem with our plan of blowing up the D'_i in arbitrary order is that over a rank 2 node of D_i both branches of D' could get blown up at the same time if this node is on only one irreducible component D'_{i_0}. This would not lead to a desingularization over rank 2 nodes. With our assumption that Y has rank 1 over all nodes of irreducible components of D we avoid this problem and obtain a smoothing $\widetilde{Y}$ of Y.

It remains to describe the geometry of the fibers of $\sigma \colon \widetilde{Y} \to Y$. We start by looking at fibers over closed points. For this we consider the normal forms of Proposition 6.7.

$\boxed{\text{Case 1.}}$ The normal form of Y around a smooth point of D is

$$x^2 + sty^2 - z^2 = 0.$$

In these local coordinates $D' = \{s = t = x = z = 0\}$. The Hessian matrix of second derivatives of this normal form is

$$\begin{pmatrix} 0 & y^2 & 0 & 0 & 2ty & 0 \\ y^2 & 0 & 0 & 0 & 2sy & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 2ty & 2sy & 0 & 0 & 2st & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

At $(0:0:0:0:1:0) \in D'$ this matrix has rank 4. Therefore the fiber of σ over this point is a $\mathbb{P}^1 \times \mathbb{P}^1$.

$\boxed{\text{Case 2.}}$ The normal form of Y around a singular rank 2 point of D is

$$x^2 + s(s+tu)y^2 - z^2 = 0.$$

The curve D' consists of two lines that intersect in the point

$$y = (0:0:0:0:1:0) \in D'.$$

We blow up in two steps

$$\widetilde{Y} \xrightarrow{\sigma_2} Y' \xrightarrow{\sigma_1} Y$$

with σ_1 blowing up one of the lines and σ_2 blowing up the strict transform of the other line.

The Hessian matrix of the above normal form has rank 3 in y. Therefore the fiber of σ_1 over y is a quadric cone C. Now, the strict transform of the other line intersects this quadric cone in one point. After a coordinate change, Y' has the same normal form as Case 1 above. Therefore the Hessian matrix at the intersection point y' of C with the strict transform of the second line has rank 4. So the fiber of σ_2 over y' is a $\mathbb{P}^1 \times \mathbb{P}^1$. The fiber of $\sigma = \sigma_2 \circ \sigma_1$ over y consists then of the strict transform of the quadric cone C under σ_2 and a $\mathbb{P}^1 \times \mathbb{P}^1$.

$\boxed{\text{Case 3.}}$ The normal form of Y around a singular rank 1 point of D is

$$x^2 + 2syz + (ty + uz)^2 = 0.$$

The curve D' consists again of two lines, but this time these lines do not intersect. Over the singular point of D we have therefore 2 points on D', namely

$$y = (0:0:0:0:0:1) \quad \text{and} \quad y' = (0:0:0:0:1:0).$$

The Hessian matrix of the normal form above has rank 4 in each point and therefore the fiber of σ is $\mathbb{P}^1 \times \mathbb{P}^1$ in both cases.

It remains now to consider the fibers over components of D'. By the above calculations the fibers over smooth points of D' are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

The fibers over each curve component of D' are therefore birational to $\mathbb{P}^1 \times \mathbb{P}^1$-bundles. By Tsen's theorem, these $\mathbb{P}^1 \times \mathbb{P}^1$ bundles are Zariski locally trivial over the components D_i', so we conclude using Proposition 6.2. ☐

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