EXCEPTIONAL COLLECTIONS OF LINE BUNDLES ON PROJECTIVE HOMOGENEOUS VARIETIES

ALEXEY ANANYEVSKIY, ASHER AUEL, SKIP GARIBALDI AND KIRILL ZAINOULLINE

Abstract. We construct new examples of exceptional collections of line bundles on the variety of Borel subgroups of a split semisimple linear algebraic group G of rank 2 over a field. We exhibit exceptional collections of the expected length for types A_2 and $B_2 = C_2$ and prove that no such collection exists for type G_2. This settles the question of the existence of full exceptional collections of line bundles on projective homogeneous G-varieties for split linear algebraic groups G of rank at most 2.

Contents

Introduction

The existence question for full exceptional collections in the bounded derived category of coherent sheaves $\mathsf{D}^{\mathrm{b}}(X)$ of a smooth projective variety X goes back to the foundational results of Beĭlinson [Beĭ78], [Beĭ84] and Bernšteĭn–Gelfand–Gelfand [BGG78] for $X = \mathbb{P}^n$. The works of Kapranov [Kap83], [Kap84], [Kap86], [Kap88] suggested that the structure of projective homogeneous variety on X should imply the existence of full exceptional collections.

Conjecture. *Let X be a projective homogeneous variety of a split semisimple linear algebraic group G over a field of characteristic zero. Then there exists a full exceptional collection of vector bundles in $\mathsf{D}^{\mathrm{b}}(X)$.*

2010 *Mathematics Subject Classification.* Primary 14F05; Secondary 14M15, 17B22, 20G15.

`Version of July 31, 2012.`

This conjecture remains largely unsolved; see [KP11, §1.1] for a recent survey of known results. The bounded derived category $\mathsf{D}^{\mathrm{b}}(X)$ has come to be understood as a homological replacement for the variety X; exceptional collections provide a way to break up $\mathsf{D}^{\mathrm{b}}(X)$ into simple components. Such decompositions of the derived category can be seen as analogous to decompositions of the motive of X, a relationship that has been put into a conjectural framework by Orlov [Orl05]. As an example, the existence of a full exceptional collection implies a splitting of the Chow motive $M(X)_{\mathbb{Q}}$ into tensor powers of Lefschetz motives (see [MM12]); this motivic decomposition is already known for projective homogeneous varieties of split linear algebraic groups (see [Köc91]).

Let X be a variety over a field k. An object E of $\mathsf{D}^{\mathrm{b}}(X)$ is called *exceptional* if $\mathrm{Ext}^*(E,E) = k$, cf. [GR87, Def. 1.1], [Bon89, §2]. Let W be a finite set and let $\mathcal{P}$ be a partial order on W. An ordered set (with respect to $\mathcal{P}$) of exceptional objects $\{E_w\}_{w\in W}$ in $\mathsf{D}^{\mathrm{b}}(X)$ is called a *$\mathcal{P}$-exceptional collection* if

$$\mathrm{Ext}^*(E_w, E_{w'}) = 0 \text{ for all } w <_{\mathcal{P}} w'.$$

If $\mathcal{P}$ is a total order, then a $\mathcal{P}$-exceptional collection is simply called an *exceptional collection.* A $\mathcal{P}$-exceptional collection $\{E_w\}_{w\in W}$ is called *full* if the smallest triangulated category containing $\{E_w\}_{w\in W}$ is $\mathsf{D}^{\mathrm{b}}(X)$ itself. Finally, a $\mathcal{P}$-exceptional collection of vector bundles $\{E_w\}_{w\in W}$ is said to be of the *expected length* if the classes $\{[E_w]\}_{w\in W}$ form a generating set of $K_0(X)$ of minimal cardinality. If $K_0(X)$ is a free abelian group (which is the case for all projective homogeneous varieties), then an exceptional collection is of the expected length if and only if its cardinality is the rank of $K_0(X)$. Note that any full $\mathcal{P}$-exceptional collection of vector bundles is of the expected length. It is expected that the converse holds [Kuz09, Conj. 9.1] for exceptional collections.

In the present paper we address the following closely related question:

Question. *Let X be the variety of Borel subgroups of a split semisimple linear algebraic group G and fix a partial order $\mathcal{P}$ on the Weyl group W of G. Does $\mathsf{D}^{\mathrm{b}}(X)$ have a $\mathcal{P}$-exceptional collection of the expected length consisting of line bundles?*

We remark that the conjecture is resolved for the varieties of Borel subgroups of split groups of classical type (see [Sam07]) and type G_2. On the one hand, the question strengthens the conjecture by requiring the collection to consist of line bundles. On the other hand, it weakens the conjecture by allowing partial orders (such as the weak or strong Bruhat orders) instead of a total order and allowing the collection to merely generate $K_0(X)$. Partially ordered exceptional collections of vector bundles are considered in [Böh06]; also see [Kan09] for an alternate approach using Frobenius splitting in finite characteristic.

So far, a natural way to propagate known exceptional collections of line bundles is to use the result of Orlov [Orl93, Cor. 2.7], that $\mathsf{D}^{\mathrm{b}}(X)$ has a full exceptional collection of line bundles if there exists a (Zariski locally trivial) projective bundle $X \to Y$ such that $\mathsf{D}^{\mathrm{b}}(Y)$ has a full exceptional collection of line bundles. More generally, $\mathsf{D}^{\mathrm{b}}(X)$ has a full exceptional collection of line bundles if X is the total space of a smooth Zariski locally trivial fibration, whose fiber and base both have derived categories with full exceptional collections of line bundles (see [CRM11]). We remark that the result of Orlov on semiorthogonal decompositions of projective bundles (hence that of Beĭlinson and Bernšteĭn–Gelfand–Gelfand on $\mathbb{P}^n$) holds over an arbitrary noetherian base scheme, see Walter [Wal03, §11]. Using such techniques, one immediately answers the question positively for all projective homogeneous varieties X for split semisimple groups of rank 2, except in the following cases: type $B_2 = C_2$ and X a 3-dimensional quadric; and type G_2 and all X. These exceptions can be viewed as key motivating examples for this paper.

It is not necessarily expected that a full $\mathcal{P}$-exceptional collection of line bundles of the expected length exists for every projective homogeneous variety. For instance, certain Grassmannians of type A_n have full exceptional collections of vector bundles, but not of line bundles. This distinction is highlighted in [CRM11, Problem 1.2] and [Sam07, Rem. 1]. The nonexistence of full exceptional collections of line bundles has also been considered in the context of providing counterexamples to King's conjecture, see [HP06], [HP11]. Observe that if $K_0(X)$ is not generated by line bundles, then $\mathsf{D}^{\mathrm{b}}(X)$ cannot possess a $\mathcal{P}$-exceptional collection of the expected length consisting of line bundles for obvious reasons. For certain minimal projective homogeneous varieties, we prove such nongeneration by line bundles results for K_0 in Propositions 4.2 and 4.4. For Borel varieties $X = G/B$, however, this observation does not help, as $K_0(X)$ is always generated by line bundles. Finally, $\mathcal{P}$-exceptional collections may exist, while exceptional collections may not.

We will now outline our main results. The paper consists of two parts. In Part I, using K_0 techniques, we introduce a new purely combinatorial algorithm for constructing $\mathcal{P}$-exceptional collections of line bundles based on a different description of the Steinberg basis [Ste75] obtained in [Ana12]. We apply it to show the following:

A. **Theorem.** *Let X be the variety of Borel subgroups of a split semisimple linear algebraic group G of rank 2 over a field of characteristic zero. Then $\mathsf{D}^{\mathrm{b}}(X)$ has a $\mathcal{P}$-exceptional collection of the expected length consisting of line bundles, for $\mathcal{P}$ a partial order isomorphic to the left weak Bruhat order on the Weyl group of G.*

For example, $\mathsf{D}^{\mathrm{b}}(X)$ possesses such a $\mathcal{P}$-exceptional collection of line bundles for the variety X of Borel subgroups of split groups of type $B_2 = C_2$ and G_2.

In §4, using combinatorial and geometric arguments we settle the question of the existence of full exceptional collections of line bundles on projective homogeneous G-varieties for every split semisimple G of rank ≤ 2 over an arbitrary field. The crux case here is that of type G_2, to which the entirety of Part II is devoted:

B. **Theorem.** *None of the three non-trivial projective homogeneous varieties of a simple algebraic group of type G_2 have an exceptional collection of the expected length consisting of line bundles.*

Acknowledgments. We are grateful to Alexander Kuznetsov for comments and for explaining us some facts concerning exceptional collections. This research was partially supported by RFBR grants 10-01-00551-a and 12-01-31100 and the Chebyshev Laboratory, Department of Mathematics and Mechanics, St. Petersburg State University under RF Government grant 11.G34.31.0026 (Ananyevskiy); NSF grant DMS-0903039 (Auel); NSA grant H98230-11-1-0178 and the Charles T. Winship Fund (Garibaldi); and NSERC Discovery 385795-2010, Accelerator Supplement 396100-2010, and the Early Researcher Award grants (Zainoulline).

Part I. Preliminaries and existence results

1. Weights and line bundles

In the present section we recall several basic facts concerning root systems, weights, associated line bundles, and the Grothendieck group K_0; see [Bou05], [Dem74], [FH91], [Pan94].

1.1. Let G be a split simple simply connected linear algebraic group of rank n over a field k. We fix a split maximal torus T and a Borel subgroup B such that $T \subset B \subset G$.

Let Λ be the weight lattice of the root system Φ of G. Observe that Λ is the group of characters of T. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a set of simple roots and let $\{\omega_1, \dots, \omega_n\}$ be the respective set of fundamental weights (a basis of Λ), i.e., $\alpha_i^\vee(\omega_j) = \delta_{ij}$. Let Φ^+ denote the set of all positive roots and let Λ^+ denote the cone of dominant weights.

1.2. Consider the integral group ring $\mathbb{Z}[\Lambda]$; its elements are finite linear combinations $\sum_i a_i e^{\lambda_i}$, $\lambda_i \in \Lambda$, $a_i \in \mathbb{Z}$. Observe that $\mathbb{Z}[\Lambda]$ can be identified with the representation ring of T. Let $X = G/B$ denote the variety of Borel subgroups of G, i.e., the variety of subgroups conjugate to B.

Consider the characteristic map for K_0,

$$\mathfrak{c}\colon \mathbb{Z}[\Lambda] \to K_0(X)$$

defined by sending e^λ to the class of the associated homogeneous line bundle $\mathscr{L}(\lambda)$ over X; see [Dem74, §2.8]. It is a surjective ring homomorphism with kernel generated by augmented invariants. More precisely, if $\mathbb{Z}[\Lambda]^W$ denotes the subring of W-invariant elements and $\epsilon\colon \mathbb{Z}[\Lambda] \to \mathbb{Z}$, $e^\lambda \mapsto 1$ is the augmentation map, then $\ker \mathfrak{c}$ is generated by elements $x \in \mathbb{Z}[\Lambda]^W$ such that $\epsilon(x) = 0$. In particular, the Picard group $\operatorname{Pic}(X)$ coincides with the set of homogeneous line bundles $\{\mathscr{L}(\lambda)\}_{\lambda \in \Lambda}$.

1.3. The Weyl group W acts linearly on Λ via simple reflections s_i as

$$s_i(\lambda) = \lambda - \alpha_i^\vee(\lambda)\alpha_i, \quad \lambda \in \Lambda.$$

Let ρ denote the half-sum of all positive roots; it is also the sum of the fundamental weights [Bou02, VI.1.10, Prop. 29].

Following [Ana12], for each $w \in W$ consider the cones Λ^+ and $w^{-1}\Lambda^+$. Let H_α denote the hyperplane orthogonal to a positive root $\alpha \in \Phi^+$. We say that H_α separates Λ^+ and $w^{-1}\Lambda^+$ if

$$\Lambda^+ \subset \{\lambda \in \Lambda \mid \alpha^\vee(\lambda) \geq 0\} \text{ and } w^{-1}\Lambda^+ \subset \{\lambda \in \Lambda \mid \alpha^\vee(\lambda) \leq 0\}.$$

or, equivalently, if $\alpha^\vee(w^{-1}\rho) < 0$. Let H_w denote the union of all such hyperplanes, i.e.,

$$H_w = \bigcup_{\alpha^\vee(w^{-1}\rho)<0} H_\alpha.$$

Consider the set $A_w = w^{-1}\Lambda^+ \setminus H_w$ consisting of weights $\lambda \in w^{-1}\Lambda^+$ separated from Λ^+ by the same set of hyperplanes as $w^{-1}\rho$. By [Ana12, Lem. 6] there is a unique element $\lambda_w \in A_w$ such that for each $\mu \in A_w$ we have $\mu - \lambda_w \in w^{-1}\Lambda^+$. In fact, the set A_w can be viewed as a cone $w^{-1}\Lambda^+$ shifted to the vertex λ_w.

1.4. **Example.** In particular, for the identity $1 \in W$ we have $\lambda_1 = 0$. Let $w = s_j$ be a simple reflection, then

$$w^{-1}\Lambda^+ = \mathbb{N}_0 s_j(\omega_j) \oplus \bigoplus_{i \neq j} \mathbb{N}_0\omega_i = \mathbb{N}_0(\omega_j - \alpha_j) \oplus \bigoplus_{i \neq j} \mathbb{N}_0\omega_i$$

and $A_w = (\omega_j - \alpha_j) + w^{-1}\Lambda^+$. Hence, in this case we have $\lambda_w = \omega_j - \alpha_j$.

For the longest element $w_0 \in W$ we have $\lambda_{w_0} = -\rho$.

1.5. By [Ana12, Thm. 2], the integral group ring $\mathbb{Z}[\Lambda]$ is a free $\mathbb{Z}[\Lambda]^W$-module with the basis $\{e^{\lambda_w}\}_{w \in W}$. As there is an isomorphism

$$\mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}[\Lambda]^W} \mathbb{Z} = \mathbb{Z}[\Lambda]/\ker \mathfrak{c} \simeq K_0(X),$$

the classes of the associated homogeneous line bundles $\mathfrak{c}(e^{\lambda_w}) = [\mathscr{L}(\lambda_w)]$, for $w \in W$, form a generating set of $K_0(X)$ of minimal cardinality.

2. $\mathcal{P}$-exceptional collections and scalar extension

In this section, we assemble some results concerning the interaction between flat base change and semiorthogonal decompositions in order to reduce questions concerning exceptional collections on smooth projective varieties over k to over the algebraic closure $\overline{k}$. If X is a variety over k, write $\overline{X}$ for the base change $X \times_k \overline{k}$, and similarly for complexes of sheaves on X.

Let W be an finite set and $\mathcal{P}$ be a partial order on W.

2.1. **Proposition.** *Let X be a smooth projective variety over a field k and let $\{E_w\}_{w\in W}$ be a $\mathcal{P}$-ordered set of line bundles on X. Then $\{E_w\}$ is a $\mathcal{P}$-exceptional collection of $\mathsf{D}^{\mathrm{b}}(X)$ if and only if $\{\overline{E}_w\}$ is a $\mathcal{P}$-exceptional collection of $\mathsf{D}^{\mathrm{b}}(\overline{X})$. Moreover, $\{E_w\}$ is full if and only if $\{\overline{E}_w\}$ is full.*

Proof. For any coherent sheaves E and F on X, we have $\mathrm{Ext}^*(E,F)\otimes_k \overline{k} \cong \mathrm{Ext}^*(\overline{E},\overline{F})$ by flat base change. In particular, E is an exceptional object of $\mathsf{D}^{\mathrm{b}}(X)$ if and only if $\overline{E}$ is an exceptional object of $\mathsf{D}^{\mathrm{b}}(\overline{X})$. Also, for each $w <_{\mathcal{P}} w'$, we have that $\mathrm{Ext}^*(\overline{E}_w,\overline{E}_{w'}) = 0$ if and only if $\mathrm{Ext}^*(E_w,E_{w'}) = 0$. Thus $\{E_w\}$ is a $\mathcal{P}$-exceptional collection of $\mathsf{D}^{\mathrm{b}}(X)$ if and only if $\{\overline{E}_w\}$ is a $\mathcal{P}$-exceptional collection of $\mathsf{D}^{\mathrm{b}}(\overline{X})$.

Suppose that $\{E_w\}$ is a full $\mathcal{P}$-exceptional collection of $\mathsf{D}^{\mathrm{b}}(X)$. As X is smooth, $\mathsf{D}^{\mathrm{b}}(X)$ is equivalent to the derived category $\mathsf{D}^{\mathrm{perf}}(X)$ of perfect complexes on X, and similarly for $\overline{X}$. As $\{E_w\}$ are line bundles, they are perfect complexes and $\{E_w\}$ is a $\mathcal{P}$-exceptional collection of $\mathsf{D}^{\mathrm{perf}}(X)$. The main results of [Kuz11] imply that $\{\overline{E}_w\}$ is a full $\mathcal{P}$-exceptional collection of $\mathsf{D}^{\mathrm{perf}}(\overline{X}) = \mathsf{D}^{\mathrm{b}}(\overline{X})$. Indeed, $\mathrm{Spec}\,\overline{k} \to \mathrm{Spec}\,k$ is faithful and the proof in [Kuz11, Prop. 5.1] immediately generalizes to the $\mathcal{P}$-exceptional setting, showing that $\{\overline{E}_w\}$ is a full $\mathcal{P}$-exceptional collection.

Now, suppose that $\{\overline{E}_w\}$ is a full $\mathcal{P}$-exceptional collection of $\mathsf{D}^{\mathrm{b}}(\overline{X})$. Let E be the triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(X)$ generated by the $\mathcal{P}$-exceptional collection $\{E_w\}$ and J be the orthogonal complement of E, i.e., there is a semiorthogonal decomposition $\mathsf{D}^{\mathrm{b}}(X) = \langle \mathsf{E}, \mathsf{J}\rangle$. By [Kuz08, Prop. 2.5], the projection functor $\mathsf{D}^{\mathrm{b}}(X) = \mathsf{D}^{\mathrm{perf}}(X) \to \mathsf{J}$ has finite cohomological amplitude, hence by [Kuz11, Thm. 7.1], is isomorphic to a Fourier–Mukai transform for some object K in $\mathsf{D}^{\mathrm{b}}(X\times_k X)$. However, $\overline{\mathsf{J}} = 0$ by assumption, hence $\overline{K} = 0$. This implies that $K = 0$, otherwise, K would have a nonzero homology group, which would remain nonzero over $\overline{k}$ by flat base change. Thus $\mathsf{J} = 0$ and so $\{E_w\}$ generates $\mathsf{D}^{\mathrm{b}}(X)$. □

In this paper, we are concerned with the case where X is a projective homogeneous variety under a linear algebraic group G that is split semisimple or has type G_2. Under this hypothesis, the pull back homomorphism $K_0(X) \to K_0(\overline{X})$ is an isomorphism [Pan94]. Together with Proposition 2.1, this reduces the question of (non)existence of an exceptional collection of expected length on X to the same question on $\overline{X}$.

3. $\mathcal{P}$-exceptional collections on Borel varieties

Consider the variety X of Borel subgroups of a split semisimple simply connected linear algebraic group G over a field k. Let $\mathscr{L}(\lambda)$ be the homogeneous line bundle over X associated to the weight λ. Recall that λ is *singular* if $\alpha^\vee(\lambda) = 0$ for some root α.

3.1. **Proposition.** *Let W be a finite set endowed with a partial order $\mathcal{P}$ and let $\{\lambda_w\}_{w\in W}$ be a set of weights indexed by W. The statements:*

(1) *$\{\mathscr{L}(\lambda_w)\}_{w\in W}$ is a $\mathcal{P}$-exceptional collection of line bundles on the Borel variety X.*
(2) *$\lambda_{w'} - \lambda_w + \rho$ is a singular weight for every $w <_{\mathcal{P}} w'$.*

are equivalent if $\mathrm{char}\, k = 0$. *If* $\mathrm{char}\, k > 0$*, then* (1) *implies* (2).

Proof. Since X is smooth, proper, and irreducible, we have $\mathrm{Hom}(\mathscr{L},\mathscr{L}) = k$ for any line bundle $\mathscr{L}$ over X. Furthermore, for any weights λ, λ', we have

$$\mathrm{Ext}^i(\mathscr{L}(\lambda),\mathscr{L}(\lambda')) = H^i(X,\mathscr{L}(\lambda)^\vee \otimes \mathscr{L}(\lambda')) = H^i(X,\mathscr{L}(\lambda' - \lambda)). \tag{3.2}$$

In particular, since 0 is a dominant weight, Kempf's vanishing theorem [Jan03, Prop. II.4.5] implies that $\mathscr{L}(\lambda)$ is an exceptional object for every weight λ.

Equation (3.2) says that (1) is equivalent to: $H^*(X,\mathscr{L}(\lambda_{w'} - \lambda_w)) = 0$ for all $w <_{\mathcal{P}} w'$. If $\mathrm{char}\, k = 0$, this is equivalent to (2) by the Borel–Weil–Bott Theorem as in [FH91, p.392] or [Jan03, II.5.5]. If $\mathrm{char}\, k = p > 0$, then X and every line bundle $\mathscr{L}(\mu)$ is defined over the field $\mathbb{F}_p$ and can be lifted to a line bundle over $\mathbb{Q}$, via a smooth projective model $\mathcal{X} \to \mathrm{Spec}\,\mathbb{Z}$ defined in terms of the corresponding Chevalley group schemes. Semicontinuity of cohomology shows that $H^i(\mathcal{X} \times_{\mathbb{Z}} \mathbb{F}_p, \mathscr{L}(\lambda_{w'} - \lambda_w)) = 0$ implies the vanishing of the analogous cohomology group over $\mathbb{Q}$, which implies (2) by the characteristic zero case. □

In the statement of Proposition 3.1, (2) need not imply (1) if $\mathrm{char}\, k > 0$. For example, it fails for $G = \mathrm{SL}_3$ over every field of finite characteristic, see [Gri80, Cor. 5.1].

3.3. **Definition.** A collection of weights $\{\lambda_w\}_{w\in W}$ is called $\mathcal{P}$*-exceptional* (resp. of the *expected length*) if the corresponding collection of line bundles $\{\mathscr{L}(\lambda_w)\}_{w\in W}$ is thus.

The proof of Theorem A consists of two steps. First, we find a maximal $\mathcal{P}$-exceptional subcollection of weights among the weights λ_w constructed in §1.3. This is done by direct computations using Proposition 3.1(2). Then we modify the remaining weights to fit in the collection, i.e., to satisfy Proposition 3.1(2) and to remain a basis. This last point is guaranteed, since we modify the weights according to the following fact.

3.4. **Lemma.** *Let $\mathcal{B}$ be a basis of $\mathbb{Z}[\Lambda]$ over $\mathbb{Z}[\Lambda]^W$ and let $e^\lambda \in \mathcal{B}$ be such that for some W-invariant set $\{\lambda_1, \lambda_2, ..., \lambda_k\}$ we have $e^{\lambda+\lambda_i} \in \mathcal{B}$ for all $i < k$ and $e^{\lambda+\lambda_k} \notin \mathcal{B}$. Then the set*

$$\big(\mathcal{B} \cup \{e^{\lambda+\lambda_k}\}\big) \smallsetminus \{e^{\lambda+\lambda_1}\}$$

is also a basis of $\mathbb{Z}[\Lambda]$ over $\mathbb{Z}[\Lambda]^W$.

Proof. Indeed, there is a decomposition

$$e^{\lambda+\lambda_k} = (e^{\lambda_1} + e^{\lambda_2} + \cdots + e^{\lambda_k})e^\lambda - e^{\lambda+\lambda_1} - e^{\lambda+\lambda_2} - \cdots - e^{\lambda+\lambda_{k-1}}$$

with the coefficients from $\mathbb{Z}[\Lambda]^W$ and an invertible coefficient at $e^{\lambda+\lambda_1}$. □

We can give a geometric description of this fact. For instance, in type B_2, the rule says that if we have a square (shifted orbit of a fundamental weight) where the center and three vertices are the basis weights, then replacing one of these basis weights by the missing vertex gives a basis; see Figure 1. For G_2 we use a hexagon (shifted orbit of a fundamental weight) instead of the square, where the center and all but one vertex are the basis weights.

Proof of Theorem A. We use the following notation: a product of simple reflections $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ is denoted by $[i_1, i_2, \ldots, i_k]$; the identity is denoted by $[]$. Given a presentation $\lambda = a_1\omega_1 + \cdots + a_n\omega_n$ in terms of fundamental weights, we denote λ by $(a_1, \ldots, a_n)$. We write $\mathcal{P}$ for the left weak Bruhat order on the Weyl group W and remark that (at least in rank 2) this partial order is isomorphic to its dual partial order.

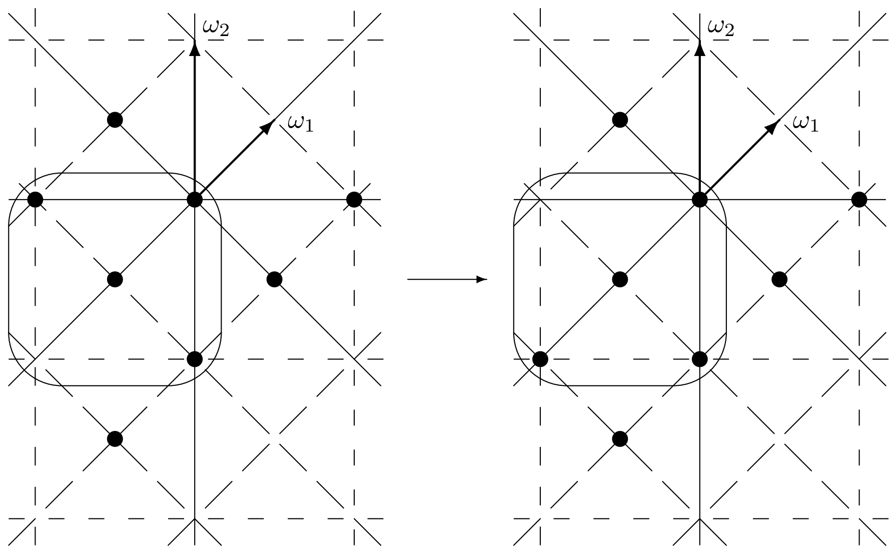

FIGURE 1. Example of the substitution described in §3.4 in the B_2 weight lattice. The thick points represent the basis, and the solid lines are the walls of Weyl chambers.

Type A_2: The Weyl group W of type A_2 consists of the following elements:

$$W = \{[], [1], [2], [2,1], [1,2], [1,2,1]\}.$$

The respective basis weights $\{\lambda_w\}_{w\in W}$ of §1.3 are given by (here the i-th weight is indexed by the i-th element of the Weyl group):

$$\{(0,0), (-1,1), (1,-1), (-1,0), (0,-1), (-1,-1)\}. \tag{3.5}$$

Direct computations using Proposition 3.1(2) show that $\{\lambda_w\}_{w\in W}$ is a $\mathcal{P}$-exceptional collection; it is of the expected length by construction.

Using Lemma 3.4, we can modify the weights $\{\lambda_w\}$ to obtain the following exceptional collection of weights (there are no Ext's from left to the right):

$$\{(0,0), (-1,0), (-2,0), (1,-1), (0,-1), (-1,-1)\}. \tag{3.6}$$

Type $A_1 \times A_1$: The Weyl group of type $A_1 \times A_1$ is the Klein four-group; the root system has orthogonal simple roots α_1, α_2 which may be taken to have square-length 2 and equal the fundamental weights. The procedure from §1.3 gives the list of basis weights $0, -\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2$, which is a full exceptional collection.

Type B_2 or C_2: The Weyl group W consists of the following elements:

$$W = \{[], [1], [2], [2,1], [1,2], [1,2,1], [2,1,2], [1,2,1,2]\}.$$

The respective basis weights $\{\lambda_w\}_{w\in W}$ of §1.3 are given by:

$$\{(0,0), (-1,1), (2,-1), (-2,1), (1,-1), (-1,0), (0,-1), (-1,-1)\}$$

(here $\omega_1 = (e_1 + e_2)/2$ and $\omega_2 = e_2$). Direct computations show that $\{\lambda_w\}_{w\in W}$ is a $\mathcal{P}$-exceptional collection, except for the weight $\lambda_{[2,1]} = (-2,1)$: indeed, the property in

Proposition 3.1(2) fails only for the weights 0 and $(-2,1)$, i.e., $\lambda_{[2,1]}+\rho$ is regular or, equivalently, $[1]*\lambda_{[2,1]}=0$ is dominant.

We modify the basis weights using Lemma 3.4: The element $e^{(-2,0)}$ has the following representation with respect to the initial basis:

$$e^{(-2,0)} = (e^{(1,0)}+e^{(1,-1)}+e^{(-1,1)}+e^{(-1,0)})e^{(-1,0)} - e^{(0,0)} - e^{(-2,1)} - e^{(0,-1)}$$

where $(e^{(1,0)}+e^{(1,-1)}+e^{(-1,1)}+e^{(-1,0)}) \in \mathbb{Z}[\Lambda]^W$. Hence, we can substitute $e^{(-2,1)}$ by $e^{(-2,0)}$. Figure 1 illustrates our arguments. Finally, after reindexing, we obtain a $\mathcal{P}$-exceptional collection of the expected length

$$\{(0,0),(-1,1),(2,-1),(-1,0),(1,-1),(-2,0),(0,-1),(-1,-1)\}. \tag{3.7}$$

Repeating Lemma 3.4 we obtain the following exceptional collection of weights

$$\{(1,0),(0,0),(-1,0),(-2,0),(2,-1),(1,-1),(0,-1),(-1,-1)\}. \tag{3.8}$$

Type G_2: The Weyl group W of type G_2 consists of the following 12 elements:

$$W = \{[],[1],[2],[2,1],[1,2],[1,2,1],[2,1,2],[2,1,2,1],[1,2,1,2],\\ [1,2,1,2,1],[2,1,2,1,2],[1,2,1,2,1,2]\}.$$

The respective basis weights $\{\lambda_w\}_{w\in W}$ of §1.3 are given by:

$$\{(0,0),(-1,1),(3,-1),(-3,2),(2,-1),(-2,1),(3,-2),\\ (-3,1),(1,-1),(-1,0),(0,-1),(-1,-1)\}.$$

Using Lemma 3.4, we obtain the following $\mathcal{P}$-exceptional collection of the expected length

$$\{(0,0),(-1,1),(3,-1),(-3,1),(2,-1),(-1,0),(1,-1),\\ (-2,0),(0,-1),(-3,0),(2,-2),(-1,-1)\}. \tag{3.9}$$

Indeed, the difference between the initial basis and the modified one consists in the substitution of $e^{(-3,2)}, e^{(-2,1)}, e^{(3,-2)}$ by $e^{(-3,0)}, e^{(-2,0)}, e^{(2,-2)}$. We proceed in several steps using the same reasoning as in the B_2-case. Denote by

$$A = e^{(1,0)}+e^{(2,-1)}+e^{(1,-1)}+e^{(-1,0)}+e^{(-2,1)}+e^{(-1,1)},$$
$$B = e^{(0,1)}+e^{(3,-1)}+e^{(3,-2)}+e^{(0,-1)}+e^{(-3,1)}+e^{(-3,2)},$$

the sums of the elements corresponding to the orbits of the fundamental weights. Note that $A,B\in\mathbb{Z}[\Lambda]^W$. We have the following decompositions with respect to the initial basis (coefficients belong to $\mathbb{Z}[\Lambda]^W$):

$$e^{(2,-2)} = Ae^{(1,-1)} - e^{(0,0)} - e^{(2,-1)} - e^{(3,-2)} - e^{(0,-1)} - e^{(-1,0)},$$
$$e^{(-4,2)} = Ae^{(-2,1)} - e^{(0,0)} - e^{(-1,0)} - e^{(-3,1)} - e^{(-3,2)} - e^{(-1,1)}.$$

Hence we can substitute $e^{(2,-2)}$ for $e^{(3,-2)}$ and $e^{(-4,2)}$ for $e^{(-3,2)}$. Using the decomposition

$$e^{(-2,0)} = Ae^{(-1,0)} - e^{(0,0)} - e^{(1,-1)} - e^{(-1,0)} - e^{(-3,1)} - e^{(-2,1)}$$

we substitute $e^{(-2,0)}$ for $e^{(-2,1)}$. Then, using

$$e^{(-4,1)} = Be^{(-1,0)} - e^{(-4,2)} - e^{(-1,1)} - e^{(2,-1)} - e^{(2,-2)} - e^{(-1,-1)}$$

substitute $e^{(-4,1)}$ for $e^{(-4,2)}$. At last, using

$$e^{(-3,0)} = Ae^{(-2,0)} - e^{(-1,0)} - e^{(0,-1)} - e^{(-1,-1)} - e^{(-4,1)} - e^{(-3,1)}$$

we substitute $e^{(-3,0)}$ for $e^{(-4,1)}$, obtaining the required basis. ∎

4. Full exceptional collections of line bundles in rank ≤ 2

Let X be a projective homogeneous variety for a split semisimple linear algebraic group G of rank ≤ 2. In this section, we provide the answer to the question:

(4.1) Does X have a full exceptional collection of line bundles?

If G has rank 2, we write $X_1 = G/P_1$ and $X_2 = G/P_2$ for the minimal projective homogeneous varieties corresponding to the two maximal parabolics and $X = G/B$ for the Borel variety. In type $B_2 = C_2$, we assume that the base field has characteristic $\neq 2$.

Types A_1 and A_2. In this case, G is isogenous to SL_2 or SL_3. In type A_1, the only projective homogeneous variety is $\mathbb{P}^1$; in type A_2, both X_1 and X_2 are $\mathbb{P}^2$. In these cases, the answer to (4.1) is "yes" by Beĭlinson [Beĭ78] and Bernšteĭn–Gelfand–Gelfand [BGG78] (which holds over any field, cf. [Wal03, §11]). In type A_2, the Borel variety X is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ and the answer is "yes" by [Orl93, Cor. 2.7] (which similarly holds over any field).

Type $A_1 \times A_1$. In this case, G is isogenous to $\mathrm{SL}_2 \times \mathrm{SL}_2$ and so both X_1 and X_2 are $\mathbb{P}^1$, while X is $\mathbb{P}^1 \times \mathbb{P}^1$. The answer to (4.1) is "yes" since we know the answers for projective space and products of projective spaces, as in the previous paragraph.

Type G_2. In this case, the answer is always "no" by Theorem B (which holds over any field).

Type $B_2 = C_2$. In this case, G is isogenous to SO_3 and to Sp_4. We know that X_2 is $\mathbb{P}^3$ and X is a $\mathbb{P}^1$-bundle over $\mathbb{P}^3$. In these cases, the answer is "yes" as in the type A_2 case. The variety X_1 is a 3-dimensional quadric, for which the answer is "no" in characteristic $\neq 2$ by the following (recall that a full exceptional collection generates K_0):

4.2. **Proposition.** *Let X be a smooth projective $(2n-1)$-dimensional quadric ($n \geq 2$) defined over a field of characteristic $\neq 2$. Then $K_0(X)$ is not generated by line bundles.*

Proof. We may assume that G is simply connected of type B_n; we write $\bar{G}$ for the adjoint group. Put P for a standard parabolic subgroup of G so that $X \simeq G/P$. We use the identification

$$\mathbb{Z} \otimes_{\mathbb{Z}[\Lambda]^W} \mathbb{Z}[\Lambda]^{W_P} = K_0(X)$$

where Λ is the weight lattice, W_P is the Weyl of the Levi part of P and $\mathbb{Z}[\Lambda]^{W_P}$ and $\mathbb{Z}[\Lambda]^W$ are the representation rings of P and G respectively.

For sake of contradiction, suppose $K_0(X)$ is generated by line bundles. Since $\mathrm{Pic}(X) = \mathbb{Z}$ is generated by $\mathscr{L}(\omega_1)$, where ω_1 is the first fundamental weight (numbered as in [Bou02]), all such bundles are powers of $\mathscr{L}(\omega_1)$, i.e., are of the form $\mathscr{L}(m\omega_1)$ for some $m \in \mathbb{Z}$.

There is a surjective homomorphism $\pi : K_0(X) \to K_0(\bar{G})$ induced by $\Lambda \to \Lambda/\Lambda_r = \mathbb{Z}/2\mathbb{Z} = \langle \sigma \rangle$, where Λ_r is the root lattice. According to [Zai, Example 3.6], $K_0(\bar{G}) = \mathbb{Z}[y]/(y^2 - 2y, 2^n y)$, where $y = 1 - e^{\sigma}$.

Now take the vector bundle E corresponding to the W_P-orbit of the last fundamental weight ω_n that can be identified with the orbit of the Weyl group of type B_{n-1} of the respective last fundamental weight. Observe that $\pi(\mathscr{L}(m\omega_1)) = 1$ and $\pi(E) = 2^{n-1}(1-y)$. As $\pi(E)$ cannot be a multiple of 1, this is a contradiction. □

We now provide the answer to the related question:

(4.3) Does X have a $\mathcal{P}$-exceptional collection of the expected length consisting of line bundles (for some choice of $\mathcal{P}$)?

Clearly, a positive answer to Question 4.1 implies a positive answer to Question 4.3. On the other hand, for a Borel variety of type G_2, the answer to Question 4.3 is "yes" (Theorem A) and the answer to Question 4.1 is "no" (Theorem B). Apart from this one case, the answers to Questions 4.1 and 4.3 agree for projective homogeneous varieties for a split semisimple group of rank ≤ 2; the only remaining case to be considered, the minimal projective homogeneous varieties of type G_2, is settled by the following:

4.4. **Proposition.** *Let X be a minimal projective homogeneous variety of type G_2 over a field of characteristic $\neq 2$. Then $K_0(X)$ is not generated by line bundles.*

Proof. Write $X = G/P$ for a maximal parabolic P of G. The derived group of the Levi subgroup L of P is isomorphic to SL_2. Put μ_2 for its center and consider the variety $Y = G/\mu_2$. We use the identification

$$K_0(Y) = \mathbb{Z} \otimes_{\mathbb{Z}[\Lambda]^W} \mathbb{Z}[t]/(t^2 - 1),$$

where $\mathbb{Z}[t]/(t^2-1)$ and $\mathbb{Z}[\Lambda]^W$ are the representation rings of μ_2 and G respectively. Computing the right-hand side obtain

$$K_0(Y) \cong \mathbb{Z}[t]/(t^2 - 1, 4t - 4).$$

There is a natural quotient morphism $\pi\colon Y \to X$ induced by the inclusion $\mu_2 \subset P$. Thus we have a homomorphism $\pi^*\colon K_0(X) \to K_0(Y)$.

For sake of contradiction, suppose that $K_0(X)$ is generated by line bundles. Recall that $\mathrm{Pic}(X) = \mathbb{Z}$ and is generated by $\mathscr{L}(\omega)$, where ω is the fundamental weight corresponding to the maximal parabolic P. Take the rank two vector bundle E corresponding to the W_P-orbit of any weight in the W-orbit of ω, except $\pm\omega$. Observe that $\pi^*\mathscr{L}(n\omega) = 1$ for all $n \in \mathbb{Z}$ and that $\pi^*E = 2t$. As π^*E cannot be a multiple of 1, this is a contradiction. ☐

Part II. Exceptional collection of line bundles on G_2-varieties

In this part, we study exceptional collections of line bundles on the Borel variety X of a group G of type G_2 over an arbitrary field.

4.5. **Example.** Suppose that G is split and $\operatorname{char} k = 0$. One of the projective homogeneous G-varieties is a 5-dimensional quadric Y and $X \to Y$ is a $\mathbb{P}^1$-bundle. An exceptional collection of vector bundles on Y described in [Kap86] and [Kap88] includes 5 line bundles. These yield an exceptional collection of line bundles on X of length 10 by [Orl93, Cor. 2.7].

As $K_0(X)$ is a free module of rank 12 (the order of the Weyl group), the collection provided in the preceding example is not of expected length. Nonetheless, we prove that it is "best possible":

4.6. **Theorem.** *Every exceptional collection of line bundles on the Borel variety of a group of type G_2 has length ≤ 10.*

A little experimentation shows that the exceptional collections on this variety are varied and numerous. For example, in characteristic zero, a computer search finds 160,017 maximal exceptional collections of the form $0, \lambda_2, \ldots, \lambda_n$ with all the weights lying in a disc of radius about 47 centered at $-\rho$. The proof of Theorem 4.6 will occupy the rest of the paper. For now, we note that the theorem is sufficient to prove Theorem B.

Proof of Theorem B. By Proposition 2.1, we may assume that the group G of type G_2 is split. For X the variety of Borel subgroups of G, $K_0(X)$ is isomorphic to $\mathbb{Z}^{12}$ and Theorem 4.6 implies that there does not exist an exceptional collection of the expected length consisting of line bundles.

Any other projective homogeneous variety Y for G can be displayed as the base of a $\mathbb{P}^1$-bundle $X \to Y$ and $K_0(Y)$ is a free $\mathbb{Z}$-module of rank 6. Hence any exceptional collection of the expected length consisting of line bundles on Y lifts to an exceptional collection of the expected length consisting of line bundles on X by [Orl93, Cor. 2.7] (which holds over any field, cf. [Wal03, §11]). Alternatively, if $\operatorname{char} k \neq 2$, one can apply Proposition 4.4. ☐

The group $K_0(X)$ depends neither on the base field nor on the particular group G of type G_2 under consideration. Combining this with Proposition 2.1, in order to prove Theorem 4.6 we may assume that the base field is algebraically closed and hence that G is split. Proposition 3.1 then reduces the proof to computations involving condition 3.1(2) applied to totally ordered lists of weights (which we write in ascending order). In fact, throughout this part, the hypothesis that a list of weights is exceptional can be strengthened to simply satisfying condition 3.1(2), though this distinction only matters in finite characteristic. We will use without much comment the fact that, for any exceptional collection $\lambda_1, \ldots, \lambda_n$ and any weight μ, the lists

$$\lambda_1 - \mu, \lambda_2 - \mu, \ldots, \lambda_n - \mu \quad \text{and} \quad -\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1 \tag{4.7}$$

are also exceptional collections. Thus, given an exceptional collection, we obtain another exceptional collection of the same length but with first entry $\lambda_1 = 0$. Note also the trivial fact that $\lambda_i \neq \lambda_j$ for all $i \neq j$, since $\operatorname{Ext}^*(\mathscr{L}(\lambda_i), \mathscr{L}(\lambda_i)) = k$ for every weight λ_i as in the proof of Proposition 3.1.

5. A dichotomy

The *crab* is the collection of weights λ of G_2 such that $\lambda + \rho$ is singular. The crab consists of weights lying on 6 *crab lines*. Any pair of lines meets only at $-\rho$, and $-\rho$ lies on all 6 crab lines. The weight zero is not on any crab line. See Figure 2 for a picture of the crab.

5.1. **Definition.** In Figure 2, we find 20 weights on the intersection of the crab lines and the singular lines, i.e., there are 20 weights λ such that λ and $\lambda + \rho$ are both singular. We call them *the 20 weights*.

We note for future reference that for each of the 20 weights λ, we have $||\lambda|| \leq 3\sqrt{3}$ and $\|\lambda + \rho\| \leq 3\sqrt{3}$.

5.2. **Lemma.** *Suppose* $0, b, c$ *is an exceptional collection.*

1. *If* b *and* c *lie on the same crab line, then* $c - b$ *is one of the 20 weights and it is on the singular line parallel to that crab line.*
2. *If* $c - b$ *and* c *lie on the same crab line, then* b *is one of the 20 weights and it is on the singular line parallel to that crab line.*

Proof. We prove (2) first. The weight b is in the crab because $0, b$ is exceptional. Further, $c - b = tc + (1 - t)(-\rho)$ for some $t \in \mathbb{R}$, hence $b = (1 - t)(c + \rho)$, which is singular because $0, c$ is exceptional, so b is one of the 20. If x is a nonzero vector orthogonal to $c + \rho$ and $(c - b) + \rho$, then it is also orthogonal to their difference, b, which proves (2). Then (1) is deduced from (2) via (4.7). ☐

5.3. **Corollary.** *In any exceptional collection* $0, \lambda_2, \ldots, \lambda_n$*, the distance between any pair of weights on the same crab line is at most* $3\sqrt{3}$.

Proof. Fix a crab line of interest and let $0, \lambda_2, \lambda_3, \ldots, \lambda_n$ be an exceptional collection. By restricting to a sub-list, we may assume that all of the weights $\lambda_2, \ldots, \lambda_n$ lie on that

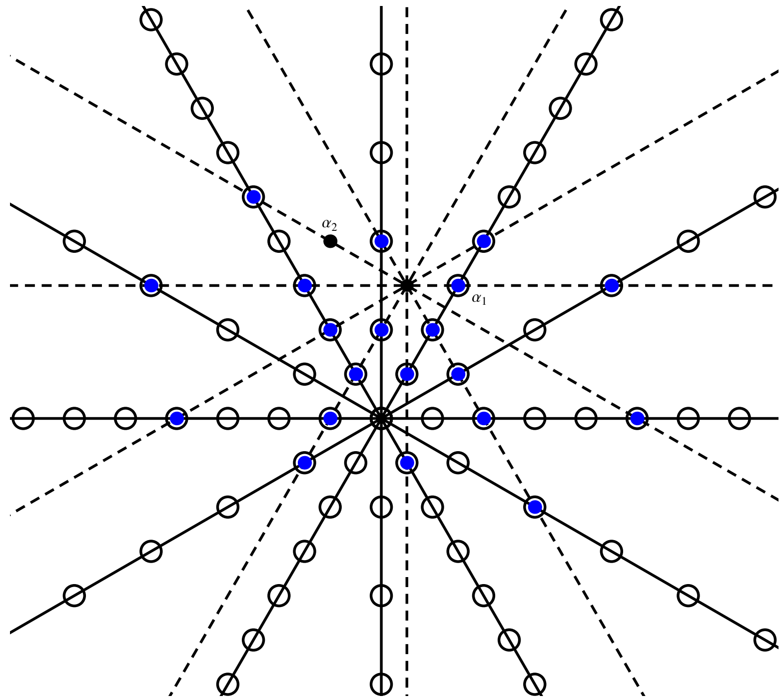

FIGURE 2. The crab lines (solid), the weights of the crab (circles), the singular lines (dashed lines), and the 20 weights (disks). All singular lines meet at 0 and all crab lines meet at $-\rho$. The simple roots are $\alpha_1 = (1, 0)$ and $\alpha_2 = (-3/2, \sqrt{3}/2)$.

crab line. By Lemma 5.2(1), $\lambda_j - \lambda_2$ is one of the 20 weights for $j = 3, \ldots, n$, hence $||\lambda_j - \lambda_2|| \leq 3\sqrt{3}$, as claimed. $\square$

5.4. **Lemma** (Trigonometry). *If two weights on crab lines are each $\geq R$ from $-\rho$ and are closer than $2(2-\sqrt{3})R$ apart, then they are on the same crab line.*

Proof. For sake of contradiction, suppose that the two weights are on different crab lines. The distance between the two weights is at least the length of the shortest line segment joining the two crab lines and meeting them at least R from $-\rho$. This segment is the third side of an isosceles triangle with two sides of length R and internal angle 2θ, hence it has length $2R\tan\theta$, where 2θ is the angle between the two crab lines. As $15^\circ \leq \theta \leq 75^\circ$, the minimum is achieved at $\tan 15^\circ = 2 - \sqrt{3}$. $\square$

The following proposition says that a close-in weight early in the exceptional collection controls the distribution of far weights on crab lines coming later in the collection.

5.5. **Proposition** (Dichotomy). *Let $0, \mu, \lambda_3, \ldots, \lambda_n$ be an exceptional collection. Then exactly one of the following holds:*

(1) *μ is one of the 20 weights, and all the λ_j such that $||\lambda_j + \rho|| > 6\sqrt{3}$ lie on the crab line parallel to the singular line containing μ.*
(2) *μ is* not *one of the 20 weights and $||\lambda_j + \rho|| < 3||\mu||$ for all $j = 3, \ldots, n$.*

Proof. First suppose that μ is *not* one of the 20 weights. As $0, \lambda_j - \mu, \lambda_j$ is an exceptional collection, $\lambda_j - \mu$ is on a crab line; by Lemma 5.2(2) it is a different crab line from λ_j. By the Trigonometry Lemma,

$$1.9\|\mu\| \geq \frac{\|\lambda_j - (\lambda_j - \mu)\|}{2(2-\sqrt{3})} \geq \min\{\|\lambda_j + \rho\|, \|\lambda_j - \mu + \rho\|\}.$$

If $\|\lambda_j - \mu + \rho\|$ is the minimum, then $\|\lambda_j + \rho\| \leq \|\lambda_j - \mu + \rho\| + \|\mu\| \leq 2.9\|\mu\|$. This proves (2).

Suppose that $0, \mu, \lambda$ is an exceptional collection such that μ is one of the 20 weights and λ does not lie on the crab line parallel to the μ singular line. Translating $0, \mu, \lambda$ by $-\mu$ gives the exceptional collection $0, \lambda - \mu$ so $\lambda - \mu$ also belongs to the crab, i.e., λ lies in the intersection of the crab and the crab shifted by μ. We will show that this implies $||\lambda + \rho|| \leq 6\sqrt{3}$, even ignoring questions of belonging to the weight lattice.

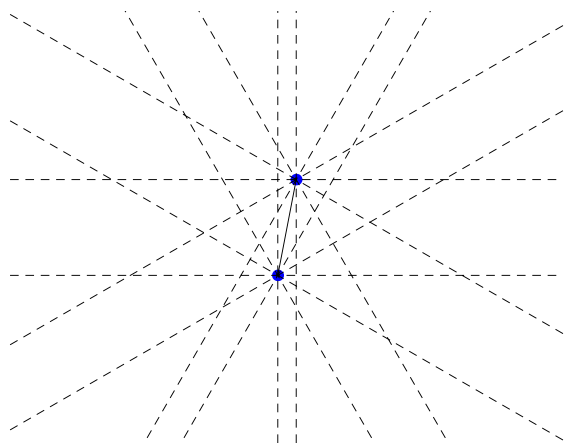

FIGURE 3. The intersection of the crab and the crab shifted by a μ on a singular line. The solid line has length $\|\mu\|$.

Indeed, the crab and the crab shifted by μ give a picture as in Figure 3. For each weight on the intersection of two dashed lines, we find a triangle where a side of length $||\mu||$ is bracketed by angles α, β that are multiples of $30°$ and $\alpha + \beta \leq 150°$. Using the Law of Sines, we find that the length of the longest of the other two sides of such a triangle is

$$\frac{\sin(\max\{\alpha, \beta\})}{\sin(180° - \alpha - \beta)} ||\mu||.$$

Plugging in all possibilities for α and β, we find that the fraction has a maximum of 2. The length of μ is at most $3\sqrt{3}$, whence the claim. □

6. Three weights on two crab lines

We now examine the possibilities for exceptional collections $0, \lambda_2, \lambda_3, \lambda_4$ such that two of the λ_j lie on one crab line and the third lies on a different crab line; there are three possible such permutations, which we label BAA, AAB, and ABA.

6.1. **Definition.** A weight a is *near* if $||a + \rho|| \leq 42$. It is *far* if $\|a + \rho\| > 42 + 3\sqrt{3}$.

6.2. **Lemma** (BAA)**.** *Suppose that $0, b, a_1, a_2$ is an exceptional collection such that a_1, a_2 are on the same crab line and b is neither on that crab line nor on the parallel singular line. Then b, a_1, a_2 are all near weights (and in fact are within 21.1 of $-\rho$).*

Proof. Translating the exceptional collection, we find the collection $0, a_1 - b, a_2 - b$, so $a_1 - b$ and $a_2 - b$ belong to the crab. Now, a_1, a_2 are on a crab line (call it A) and b is not on the parallel singular line (i.e., b is not parallel to $a_2 - a_1$), therefore $a_1 - b$ and $a_2 - b$ are not on the A crab line. Furthermore, because the direction $a_2 - a_1 = (a_2 - b) - (a_1 - b)$ characterizes the A line, we conclude that $a_1 - b$ and $a_2 - b$ lie on different crab lines.

However,

$$||(a_2 - b) - (a_1 - b)|| = ||a_2 - a_1|| \leq 3\sqrt{3}$$

by Corollary 5.3, hence

$$||a_j - b + \rho|| \leq \frac{3\sqrt{3}}{2(2-\sqrt{3})}$$

by the Trigonometry Lemma 5.4. By the triangle inequality

$$||a_j - b|| \leq ||a_j - b + \rho|| + || - \rho|| \leq \frac{3\sqrt{3}}{2(2-\sqrt{3})} + \sqrt{7} < 12.4.$$

As a_j and b are on different crab lines, the argument in the Trigonometry Lemma 5.4 gives that $||a_j + \rho||$ and $||b + \rho||$ are at most

$$\frac{\frac{3\sqrt{3}}{2(2-\sqrt{3})} + \sqrt{7}}{2(2-\sqrt{3})} < 21.04. \qquad \square$$

Here are two corollaries from the Trigonometry Lemma 5.4.

6.3. **Corollary.** *If λ and $\lambda + \rho$ are on crab lines and $||\lambda + \rho|| > 7.7$, then λ and $\lambda + \rho$ are on the same crab line.*

Proof. We use the triangle inequality to bound the distance of $\lambda + \rho$ from $-\rho$:

$$||\lambda + 2\rho|| \geq ||\lambda + \rho|| - ||-\rho|| > 7.7 - \sqrt{7} > 5.05.$$

The distance between λ and $\lambda + \rho$ is $||\rho|| = \sqrt{7} < 2(2-\sqrt{3})5.05$, so taking $R = 5.05$ in the Trigonometry Lemma 5.4 gives the claim. $\square$

6.4. **Corollary.** *If a and $b - a$ both lie on some crab line A, then so does $b + \rho$. If furthermore $||b + \rho|| > 7.7$, then b also lies on A.*

Proof. Let x be a nonzero vector orthogonal to $a+\rho$ and $b-a+\rho$. Then x is also orthogonal to $(b - a + \rho) + a + \rho = (b + \rho) + \rho$; this proves the first claim. For the second claim, we apply Corollary 6.3. $\square$

6.5. **Lemma** (AAB)**.** *If $0, a_1, a_2, b$ is an exceptional collection where a_1, a_2 are on one crab line and b is on a different crab line, then at least one of a_1, a_2, b is near.*

Proof. For sake of contradiction, suppose all three nonzero weights are at least 42 from $-\rho$. Translating, we find an exceptional collection $0, a_2 - a_1, b - a_1$, where for $j = 1, 2$ we have $||b - a_j|| > 2(2 - \sqrt{3})42$. Further,

$$||b - a_j + \rho|| \geq ||b - a_j|| - ||-\rho|| > 2(2-\sqrt{3})42 - \sqrt{7} > 19.8.$$

Now

$$||(b - a_2) - (b - a_1)|| = ||a_1 - a_2|| \leq 3\sqrt{3} < 2(2-\sqrt{3})19.8,$$

so by the Trigonometry Lemma 5.4 $b - a_2$ and $b - a_1$ lie on the same crab line.

As a_1, a_2 also lie on one crab line, we can find nonzero vectors x, y such that x is orthogonal to $b - a_j + \rho$ and y is orthogonal to $a_j + \rho$ for $j = 2, 3$. It follows that

$$a_1 - a_2 = (a_1 + \rho) - (a_2 + \rho) = (b - a_2 + \rho) - (b - a_1 + \rho)$$

is orthogonal to both x and y. As $a_1 - a_2 \neq 0$, it follows that the four weights $a_j, b - a_j$ for $j = 2, 3$ all lie on one crab line. Corollary 6.4 gives that $b + \rho$ lies on this same line. As b is also on a crab line and b is at least 42 from $-\rho$, Corollary 6.3 gives that b and $b + \rho$ are on the same crab line. This contradicts the hypothesis that a_1, b are on different crab lines. □

We now prepare for ABA, the most complicated of the three configurations.

6.6. **Definition.** The *mirror 20 weights* consist of the intersection of the crab with the crab shifted to $-\rho$. A weight μ is one of the mirror 20 weights if both μ and $\mu + \rho$ are in the crab. The lines through -2ρ parallel to the crab lines will be called the *mirror singular lines.*

Now we need a "mirror" version of Proposition 5.5(2).

6.7. **Proposition.** *If $0, \lambda, \mu$ is an exceptional collection with $\|\lambda\| \geq 2.9\|\mu + \rho\| + 7.6$, then μ is one of the mirror 20 weights on the mirror singular line parallel to λ. In particular, in this case $\|\mu + \rho\| \leq 3\sqrt{3}$.*

Proof. As λ is in the crab, $\lambda + \rho$ is singular, thus $-2\rho - \lambda + \rho = -\rho - \lambda = -(\lambda + \rho)$ is singular, hence $-2\rho - \lambda$ is in the crab. Also $\mu - \lambda$ is in the crab by exceptionality. We will show that $\mu - \lambda$ and $-2\rho - \lambda$ are on the same crab line, hence that $(\mu - \lambda) - (-2\rho - \lambda + \rho) = \mu + \rho$ is also on the crab and thus μ is one of the mirror 20 weights.

We have

$$\|(\mu - \lambda) - (-2\rho - \lambda)\| = \|\mu + 2\rho\| \leq \|\mu + \rho\| + \sqrt{7}$$

by the triangle inequality. By the Trigonometry Lemma 5.4, μ will then be one of the mirror 20 weights as long as $\mu - \lambda$ and $-2\rho - \lambda$ are a distance

$$\frac{\|\mu + \rho\| + \sqrt{7}}{2(2 - \sqrt{3})} < 1.87\|\mu + \rho\| + 4.94$$

from $-\rho$. But indeed, by hypothesis, we have

$$\|\mu - \lambda + \rho\| \geq \|\lambda\| - \|\mu + \rho\| \geq 1.9\|\mu + \rho\| + 7.6$$

and

$$\|-2\rho - \lambda + \rho\| \geq \|\lambda\| - \|\rho\| \geq 2.9\|\mu + \rho\| + 7.6 - \sqrt{7} > 2.9\|\mu + \rho\| + 4.96.$$

The final claims are apparent. □

6.8. **Lemma** (ABA)**.** *Suppose that $0, a_1, b, a_2$ is an exceptional collection of far weights such that a_1, a_2 are on the same crab line and b is on a different crab line. Then the collection $0, a_1, b, a_2$ is maximal.*

By *maximal* we mean that there is no exceptional collection $0, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ that contains $0, a_1, b, a_2$ as a sub-collection.

Proof. We have a number of cases, each of which we will deal with by contradiction.

Case 1: Consider extending the collection as $0, \mu, a_1, b, a_2$. If μ is on the same crab line as a_1, a_2, then the AAB Lemma 6.5 applied to the exceptional collection $0, \mu, a_1, b$, together with the fact that a_1, b are far, implies that μ is near. But this is impossible since μ and a_1 can be at most $3\sqrt{3}$ apart. If μ is on a different crab line than a_1, a_2, then $0, \mu, a_1, a_2$ is exceptional, which contradicts the BAA Lemma 6.2 (since a_1, a_2 are far) unless μ is on the singular line parallel to the crab line containing a_1, a_2. But $0, \mu, b$ is exceptional, μ is one of the 20 weights, and b is far, hence b is on the crab line parallel to the singular line containing μ. This is impossible since b and a_1, a_2 are on different crab lines.

Case 2: Similarly, consider extending the collection as $0, a_1, b, a_2, \mu$. If μ is on the same crab line as a_1, a_2, then the exceptionality of $0, b, a_2, \mu$ contradicts the BAA Lemma 6.2 since b, a_2 are far (b cannot be on the singular line parallel to the crab line containing a_2, μ because it is far). If μ is on a different crab line than a_1, a_2, then by the AAB Lemma 6.5 and the fact that a_1, a_2 are far, we have that μ is near. By Proposition 6.7 applied to $0, a_1, \mu$ and $0, b, \mu$, we have that μ is one of the mirror 20 weights on the mirror singular line parallel to the crab lines containing a_1 and b, hence $\mu = -2\rho$, contradicting the hypothesis that μ is in the crab.

Case 3: Now consider extending the collection as $0, a_1, \mu, b, a_2$. If μ is on the same crab line as a_1, a_2, then the AAB Lemma 6.5 applied to the exceptional collection $0, a_1, \mu, b$, together with the fact that a_1, b are far, implies that μ is near. But this is impossible since μ and a_1 can be at most $3\sqrt{3}$ apart. Similarly, if μ is on the same crab line as b, then the AAB Lemma 6.5 applied to the exceptional collection $0, \mu, b, a_2$, together with the fact that b, a_2 are far, implies that μ is near. But this is impossible since μ and b can be at most $3\sqrt{3}$ apart.

Thus μ is not on the crab line containing any of a_1, b, a_2. As in (4.7), the collection

$$0,\ a_2 - b,\ a_2 - \mu,\ a_2 - a_1,\ a_2$$

is exceptional. By Lemma 5.2, $a_2 - a_1$ is one of the 20 weights, in particular $\|a_2 - a_1 + \rho\| \leq 3\sqrt{3}$. Since b, a_2 are far and on different crab lines, $\|b + \rho\|, \|a_2 + \rho\| \geq 42 + 3\sqrt{3}$, hence by the Trigonometry Lemma 5.4, we have

$$\|a_2 - b\| \geq 2(2 - \sqrt{3})(42 + 3\sqrt{3}) > 2.9 \cdot 3\sqrt{3} + 7.6.$$

Applying Proposition 6.7 to the exceptional collection $0, a_2 - b, a_2 - a_1$, we have that $a_2 - a_1$ is one of the mirror 20 weights on the mirror singular line parallel to the crab line containing $a_2 - b$. In particular, $\|a_2 - a_1 + \rho\| \leq \sqrt{3}$.

The first option of the Dichotomy Proposition 5.5 applied to the exceptional collection $0, \mu, b, a_2$ is impossible since b, a_2 are far and on different crab lines. Hence by Dichotomy, $\|\mu\| > \frac{1}{3}(42 + 3\sqrt{3}) > 15$. In particular $\|\mu + \rho\| > 15 - \sqrt{7}$, so by the Trigonometry Lemma 5.4 (using that a_2 is far), we have $\|a_2 - \mu\| > 2(2 - \sqrt{3})(15 - \sqrt{7}) > 7 > 3\sqrt{3}$, in particular $a_2 - \mu$ is not one of the 20 weights.

It follows that the second option in Dichotomy holds for $0, a_2 - \mu, a_2$ and $42 + 3\sqrt{3} < \|a_2 + \rho\| < 3\|a_2 - \mu\|$. But then we have

$$\|a_2 - \mu\| > \frac{1}{3}(42 + 3\sqrt{3}) > 15 > 2.9 \cdot \sqrt{3} + 7.6$$

so that we can apply Proposition 6.7 to the exceptional collection $0, a_2 - \mu, a_2 - a_1$. We conclude that $a_2 - a_1$ is one of the mirror 20 weights on the mirror singular line parallel to the crab line containing $a_2 - \mu$.

Since b is far and μ is on a different crab line, the Trigonometry Lemma 5.4 says that

$$\|\mu - b\| > \frac{15 - \sqrt{7}}{2(2 - \sqrt{3})} > 3\sqrt{3},$$

so $a_2 - \mu$ and $a_2 - b$ cannot lie on the same crab line. But then it is impossible for $a_2 - a_1$ to be on the mirror singular lines parallel to the crab lines of both $a_2 - \mu$ and $a_2 - b$. Therefore no such μ can exist.

Case 4: Finally, consider extending the collection as $0, a_1, b, \mu, a_2$. If μ is far, then by interchanging the roles of μ and b, we can use the previous argument. Hence we can assume μ is not far. If μ is on the same crab line as a_1, a_2, then the exceptionality of $0, b, \mu, a_2$ contradicts the BAA Lemma 6.2 since b, a_2 are far (in particular, b cannot be on the singular line parallel to the crab line containing a_2, μ). Similarly, if μ is on the same crab line as b, then the exceptionality of $0, a_1, b, \mu$ contradicts BAA Lemma 6.2. Thus μ is not on the crab line containing any of a_1, b, a_2. If μ is one of the 20 weights, then we can apply Proposition 6.7 to the exceptional collections $0, a_1, \mu$ and $0, b, \mu$ (since $42+3\sqrt{3} \geq 2.9\cdot 3\sqrt{3}+7.6$), concluding that μ is also one of the mirror 20 weights contained on the mirror singular lines parallel to crab lines containing a_1, b, which is impossible. Hence as before, the Dichotomy Proposition 5.5 implies that $\|a_2 - \mu\| > 7$. As in (4.7), the collection $0, a_2 - \mu, a_2 - b, a_2 - a_1, a_2$ is exceptional and we can use the previous argument.

We have thus ruled out all possible exceptional extensions of $0, a_1, b, a_2$. □

Combining the Lemmas 6.2, 6.5, and 6.8 gives the following:

6.9. **Proposition.** *Suppose that a_1, a_2, b are far weights such that a_1, a_2 lie on the same crab line and b lies on a different crab line. Then:*

(1) *Neither $0, b, a_1, a_2$ nor $0, a_1, a_2, b$ are exceptional collections.*
(2) *If $0, a_1, b, a_2$ is an exceptional collection, then it is maximal.*

7. Computer calculations

Our proof of Theorem 4.6 makes use of the following concrete facts, which can be easily verified by computer:

7.1. **Fact.** Every exceptional collection $0, \lambda_2, \ldots, \lambda_n$ with all λ_j non-far has $n \leq 10$.

7.2. **Fact.** Let A be a crab line and S the set of weights on the union of the singular line and the mirror singular line parallel to A. Then every exceptional collection $0, \lambda_2, \ldots, \lambda_n$ of weights in S has $n \geq 5$. (Note that as the λ_j's belong to the crab, they are all selected from the union of the 20 weights and the mirror 20 weights.)

7.3. **Fact.** If $0, \lambda_2, \ldots, \lambda_n$ is an exceptional collection with $n = 9$ or 10, with all weights non-far, with all crab lines containing at most 2 weights, and with one crab line containing no weights, then $\|\lambda_j + \rho\| \leq 5$ for all $j = 2, \ldots, n$.

We used Mathematica to check these facts. We first wrote a function `IsSingular` that returns `True` if a weight is singular and `False` otherwise. With the following code, and lists of weights `L1` and `L2`, the command `FindCollections[L1, L2]` will fill the global variable `collections` with a list of all of the maximal exceptional collections that begin with `L1` and such that all weights following `L1` come from `L2`. In the code, `rho` denotes the highest root written in terms of the fundamental weights. We omit the sanity checks that ensure that for the initial values of `L1` and `L2`, appending each element of `L2` to `L1` results in an exceptional collection.

```
collections = {};
FindCollections[L1_, L2_] := Module[{tmpL1},
   If[Length[L2] == 0, AppendTo[collections, L1],
    Do[
     tmpL1 = Append[L1, L2[[i]]];
     FindCollections[tmpL1,
      Select[Delete[L2, i], IsSingular[# - L2[[i]] + rho] &]],
      {i, 1, Length[L2]}]]];
```

For example, to check Fact 7.1, we constructed the list `L2` consisting of all 445 non-far weights in the crab and executed `FindCollections[{{0, 0}}, L2]` to obtain the list of the 160,017 maximal exceptional collections $0, \lambda_2, \ldots, \lambda_n$ with all λ_j non-far. With this list in hand, it is not difficult to select out collections meeting the criteria of Facts 7.2 and 7.3.

8. Bounding exceptional collections

This section will complete the proof of Theorem 4.6.

8.1. **Lemma.** *In any exceptional collection $0, \lambda_2, \ldots, \lambda_n$, at most 5 of the λ_j's lie on any given crab line.*

Proof. Fix a crab line of interest and let $0, \lambda_2, \lambda_3, \ldots, \lambda_n$ be an exceptional collection. By restricting to a sub-list, we may assume that all of the weights $\lambda_2, \ldots, \lambda_n$ lie on that crab line. By Corollary 5.2, $\lambda_j - \lambda_2$ is one of the 20 weights for $j = 3, \ldots, n$, and Figure 2 shows that $\lambda_3, \ldots, \lambda_n$ has length at most 5 corresponding to having 6 weights on the line of interest, and that the proof is complete except in the case where the crab line makes a $120°$ angle with the horizontal.

For that line, we must argue that $\lambda_3 - \lambda_2, \ldots, \lambda_n - \lambda_2$ cannot be the 5 weights of the 20 depicted in the figure. Indeed, if they were, one could translate by $\lambda_3 - \lambda_2$ to transform this to an exceptional collections $0, \lambda_4 - \lambda_3, \ldots, \lambda_7 - \lambda_3$ and $\lambda_j - \lambda_3$ must belong to the crab for $j \geq 4$. But we can see from the figure that this does not happen for any of the 5 choices for $\lambda_3 - \lambda_2$, hence the claim. $\square$

8.2. **Proposition.** *If $0, \lambda_2, \ldots, \lambda_n$ is an exceptional collection containing at least 3 weights on some crab line A, then $n \leq 10$ and all the weights off A are near.*

Proof. In the exceptional collection $0, \lambda_2, \ldots, \lambda_n$, let a_1, a_2, a_3 be three weights on A. Then every nonzero weight b in the collection and off A either precedes a_2, a_3 or follows a_1, a_2.

First suppose b precedes a_2, a_3. Then by the BAA Lemma 6.2, either b, a_2, a_3 are all near or b is on the singular line parallel to A. In the latter case, b is one of the 20 weights and so is near.

Suppose that b follows a_1, a_2, then shifting by $-a_1$ gives an exceptional collection $0, a_2 - a_1, b - a_1$ where $a_2 - a_1$ is one of the 20 weights. By the Dichotomy Proposition 5.5, we have two possibilities:

<u>*Case 1:*</u> We could have that $\|b - a_1 + \rho\| \leq 6\sqrt{3}$, but in that case we find that

$$\|b - a_1\| - \|\rho\| \leq \|b - a_1 + \rho\| \leq 6\sqrt{3},$$

hence $\|b - a_1\| \leq 6\sqrt{3} + \sqrt{7}$. But b and a_1 lie on different crab lines, so by the Trigonometry Lemma 5.4, we find that $\min\{\|b+\rho\|, \|a_1+\rho\|\}$ is at most $(6\sqrt{3}+\sqrt{7})/(2(2-\sqrt{3})) < 25$. If $\|a + \rho\| < 25$, then

$$\|b + \rho\| \leq \|a_1 + \rho\| + \|b - a_1\| < 25 + 6\sqrt{3} + \sqrt{7} < 42$$

and b is near. (Note that if a_1 is non-near, then we would have $\|b + \rho\| < 25$ and this case is impossible.)

<u>*Case 2:*</u> Alternately, $b - a_1$ could lie on A. In that case, as b is not on A, Corollary 6.4 gives that $\|b + \rho\| \leq 7.7$. Thus all weights in the exceptional collection off A are near.

It remains to argue that the collection has length ≤ 10. If the weights on A are non-far, then all the weights in the collection are non-far and we are done by Fact 7.1. Therefore, we may assume that some of the weights on A are far, hence all weights on A are non-near. Let b be a nonzero weight in the collection that is off A. If it precedes a_2, a_3, then b is one of the 20 weights (because b, a_2, a_3 cannot all be near). Otherwise, b comes after a_1, a_2 and we are in Case 2 above, so $2.9\|b+\rho\|+7.6 \leq 29.93$; by Proposition 6.7, b is one of the mirror 20 weights and lies on the mirror singular line parallel to A. Fact 7.2 and Lemma 8.1 show that one cannot obtain an exceptional collection of length > 10. □

We can now conclude the proof of the main theorem.

Proof of Theorem 4.6. For sake of contradiction, we suppose we are given an exceptional collection $0, \lambda_2, \ldots, \lambda_{11}$. By Proposition 8.2, no crab line contains more than 2 weights.

We claim that the number F of far weights in the collection is 1 or 2. Indeed, by Fact 7.1, F is positive. Suppose that it is at least 3. Then by Proposition 6.9, all far weights lie on different crab lines, leaving $6-F$ crab lines for the remaining $10-F$ nonzero weights; but the remaining crab lines can only hold $12-2F$ weights, which contradicts our hypothesis that $F \geq 3$; hence $F = 1$ or 2.

We will now pick a crab line A and a subset S of $\lambda_2, \ldots, \lambda_{11}$ containing all the far weights and all the weights on A, and such that $|S| = 1$ or 2.

Case $F = 1$: If $F = 1$, we take A to be the crab line containing the far weight and let S be the set of λ_j's lying on A; by hypothesis $|S| \leq 2$.

Case $F = 2$: If both of the far weights are on one crab line, then we take it to be A and S to be the set of far weights.

Otherwise, the two far weights are on different crab liens. We claim that one of these crab lines, call it A, contains exactly one weight from the exceptional collection. Indeed, otherwise there would be two crab lines each containing two weights; as all of these are non-near by Corollary 5.3, this contradicts Proposition 6.9, verifying the claim. We take S to be the far weights in the exceptional collection.

We have found S as desired, and deleting it from the exceptional collection leaves one as in Fact 7.3 and we conclude that $\|\lambda_j + \rho\| \leq 5$ for all λ_j not in S. If such a λ_j precedes one of the far weights, then it is one of the 20 weights by the Dichotomy Proposition 5.5; if it follows one of the far weights then it is one of the mirror 20 weights by Proposition 6.7. But deleting S from our exceptional collection leaves an exceptional collection starting with 0 and containing at least 8 nonzero, non-far weights all lying off A, which contradicts Fact 7.2. □

References

[Ana12] A. Ananyevskiy, *On the algebraic K-theory of some homogeneous varieties*, Doc. Math. **17** (2012), 167–193.

[Beĭ78] A. A. Beĭlinson, *Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 68–69.

[Beĭ84] ———, *The derived category of coherent sheaves on $\mathbf{P}^n$*, Selecta Math. Soviet. **3** (1983/84), no. 3, 233–237, Selected translations.

[BGG78] I. N. Bernšteĭn, I. M. Gelfand, and S. I. Gelfand, *Algebraic vector bundles on $\mathbf{P}^n$ and problems of linear algebra*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 66–67.

[Böh06] C. Böhning, *Derived categories of coherent sheaves on rational homogeneous manifolds*, Doc. Math. **11** (2006), 261–331.

[Bon89] A. I. Bondal, *Representations of associative algebras and coherent sheaves*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 25–44.

[Bou02] N. Bourbaki, *Lie groups and Lie algebras: Chapters 4–6*, Springer-Verlag, Berlin, 2002.

[Bou05] ———, *Lie groups and Lie algebras: Chapters 7–9*, Springer-Verlag, Berlin, 2005.

[CRM11] L. Costa, S. Di Rocco, and R. M. Miró-Roig, *Derived category of fibrations*, Math. Res. Lett. **18** (2011), no. 3, 425–432.

[Dem74] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 53–88, Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I.

[FH91] W. Fulton and J. Harris, *Representation theory: a first course*, Graduate texts in mathematics, vol. 129, Springer, 1991.

[GR87] A. L. Gorodentsev and A. N. Rudakov, *Exceptional vector bundles on projective spaces*, Duke Math. J. **54** (1987), no. 1, 115–130.

[Gri80] W. L. Griffith, Jr., *Cohomology of flag varieties in characteristic p*, Illinois J. Math. **24** (1980), no. 3, 452–461.

[HP06] L. Hille and M. Perling, *A counterexample to King's conjecture*, Compos. Math. **142** (2006), no. 6, 1507–1521.

[HP11] ———, *Exceptional sequences of invertible sheaves on rational surfaces*, Compos. Math. **147** (2011), no. 4, 1230–1280.

[Jan03] J. C. Jantzen, *Representations of algebraic groups*, 2nd ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003.

[Kan09] M. Kaneda, *On the structure of parabolic Humphreys-Verma modules*, New trends in combinatorial representation theory, RIMS Kôkyûroku Bessatsu, B11, Res. Inst. Math. Sci. (RIMS), Kyoto, 2009, pp. 117–123.

[Kap83] M. M. Kapranov, *The derived category of coherent sheaves on Grassmann varieties*, Funktsional. Anal. i Prilozhen. **17** (1983), no. 2, 78–79.

[Kap84] ———, *Derived category of coherent sheaves on Grassmann manifolds*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 1, 192–202.

[Kap86] ———, *Derived category of coherent bundles on a quadric*, Funktsional. Anal. i Prilozhen. **20** (1986), no. 2, 67.

[Kap88] ———, *On the derived categories of coherent sheaves on some homogeneous spaces*, Invent. Math. **92** (1988), no. 3, 479–508.

[Köc91] B. Köck, *Chow motif and higher Chow theory of G/P*, Manuscripta Math. **70** (1991), 363–372.

[KP11] A. Kuznetsov and A. Polischchuk, *Exceptional collections on isotropic Grassmannians*, preprint arxiv:1110.5607, 2011.

[Kuz08] A. Kuznetsov, *Lefschetz decompositions and categorical resolutions of singularities*, Selecta Math. (N.S.) **13** (2008), no. 4, 661–696.

[Kuz09] ———, *Hochschild homology and semiorthogonal decompositions*, preprint math.AG/0904.4330, 2009.

[Kuz11] ———, *Base change for semiorthogonal decompositions*, Compos. Math. **147** (2011), no. 3, 852–876.

[MM12] G. Tabuada and M. Marcolli, *From exceptional collections to motivic decompositions via noncommutative motives*, preprint arXiv:1202.6297, 2012.

[Orl93] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. **41** (1993), no. 1, 133–141.

[Orl05] ———, *Derived categories of coherent sheaves, and motives*, Uspekhi Mat. Nauk **60** (2005), no. 6(366), 231–232.

[Pan94] I. A. Panin, *On the algebraic K-theory of twisted flag varieties*, K-Theory **8** (1994), no. 6, 541–585.

[Sam07] A. Samokhin, *Some remarks on the derived categories of coherent sheaves on homogeneous spaces*, J. Lond. Math. Soc. (2) **76** (2007), no. 1, 122–134.

[Ste75] R. Steinberg, *On a theorem of Pittie*, Topology **14** (1975), 173–177.

[Wal03] C. Walter, *Grothendieck-Witt groups of projective bundles*, K-theory preprint archive `http://www.math.uiuc.edu/K-theory/644/`, 2003.

[Zai] K. Zainoulline, *Twisted γ-filtration of a linear algebraic group*, to appear in *Compositio Math.*

(Alexey Ananyevskiy) Department of Mathematics and Mechanics, St. Petersburg State University, Universitetskiy Prospekt 28, St. Petersburg, 198504, Russia

(Asher Auel and Skip Garibaldi) Department of Mathematics & Computer Science, Emory University, Atlanta, GA 30307, USA

(Kirill Zainoulline) Department of Mathematics and Statistics, University of Ottawa, 585 King Edward, Ottawa ON K1N6N5, Canada