Dartmouth College Department of Mathematics
Math 81/111 Abstract Algebra
Winter 2020

Problem Set # 6 (due in class on Friday 28 February)

Notation: The **Galois group** of a polynomial $f(x)$ over a field F is defined to be the F-automorphism group of its splitting field E.

Problems:

1. Let $f(x) \in \mathbb{R}[x]$ be a cubic polynomial with discriminant Δ.

(a) You know that $\Delta = 0$ if and only if $f(x)$ has a repeated root in its splitting field. Prove that in this case, all the roots of $f(x)$ are real.

(b) Prove that $\Delta > 0$ if and only if the roots of $f(x)$ are distinct and are all real.

(c) Prove that $\Delta < 0$ if and only if $f(x)$ has a single real root and a pair of complex conjugate (nonreal) roots.

For a polynomial $f(x) \in \mathbb{R}[x]$ of degree $n \geq 1$ with discriminant $\Delta \neq 0$, state and prove a formula for the sign of Δ in terms of the number of pairs of complex conjugate (nonreal) roots.

2. Let $\gamma = \sqrt{2 + \sqrt{2}} \in \mathbb{R}$.

(a) Show that $\mathbb{Q}(\gamma)/\mathbb{Q}$ is Galois with cyclic Galois group.

(b) Show that $\mathbb{Q}(\gamma, i) = \mathbb{Q}(\zeta_{16})$ and calculate the Galois group $\mathrm{Gal}(\mathbb{Q}(\gamma, i)/\mathbb{Q})$.

3. Let K/F be a Galois extension with Galois group isomorphic to $C_2 \times C_{12}$. How many subextensions of $K/M/F$ are there satisfying:

(a) $[K : M] = 6$

(b) $[M : F] = 6$

(c) $\mathrm{Gal}(K/M)$ isomorphic to C_6

(d) $\mathrm{Gal}(M/F)$ isomorphic to C_6

4. Let p be a prime number and S_p the symmetric group on p things.

(a) Prove that an element of S_p has order p if and only if it is a p-cycle.

(b) Prove that S_p is generated by any choice of a p-cycle and a transposition. Find a composite n and a choice of an n-cycle and a transposition that do not generate S_n.

(c) Let $F \subset \mathbb{R}$ be a subfield. Prove that if $f(x) \in F[x]$ is an irreducible polynomial of degree p having $p - 2$ real roots, then the Galois group of $f(x)$ over F is isomorphic to S_p.

(d) Let $F \subset \mathbb{R}$ be a subfield. Prove that if $f(x) \in F[x]$ is an irreducible cubic polynomial with $\Delta < 0$, then the Galois group of $f(x)$ over F is isomorphic to S_3.

(e) Prove that the Galois group of the polynomial $x^3 - x - 1$ over $\mathbb{Q}$ is isomorphic to S_3.

(f) Prove that the Galois group of the polynomial $x^5 - x^4 - x^2 - x + 1$ over $\mathbb{Q}$ is isomorphic to S_5. **Hint.** You are allowed to use real analysis (e.g., the intermediate value theorem), but as a challenge, try to find a purely algebraic (possibly computer-aided) way.

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