Dartmouth College Department of Mathematics
Math 81/111 Abstract Algebra
Winter 2020

Problem Set # 5 (due in class on Friday 14 February ♡)

Notation: Let K and L be subfields of a field M. The **compositum** of K and L, denoted KL, is defined to be the smallest subfield of M containing both K and L, equivalently, the intersection of all subfields of M containing K and L. If additionally K and L are both extensions of a field F, we say that the extensions K/F and L/F are **linearly disjoint** if any F-linearly independent subset of K is L-linearly independent in KL and if any F-linearly independent subset of L is K-linearly independent in KL.

Problems:

1. Let F be a field and K/F and L/F be subextensions of a field extension M/F.

(a) Prove that if $K = F(\alpha_1, \ldots, \alpha_n)$ and $L = F(\beta_1, \ldots, \beta_m)$ are finitely generated, then $KL = F(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$.

(b) Prove that if K/F and L/F are finite then KL/F is finite and $[KL : F] \leq [K : F]\,[L : F]$ with equality if and only if K/F and L/F are linearly disjoint.
Hint. Prove that if $x_1, \ldots, x_n$ is an F-basis for K and $y_1, \ldots, y_m$ is an F-basis for L, then the products $x_i y_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ span KL/F and are an F-basis if and only if K/F and L/F are linearly disjoint.

(c) Prove that finite extensions K/F and L/F of relatively prime degree are linearly disjoint.

(d) Prove that if K/F and L/F are linearly disjoint then $K \cap L = F$. Find an example showing that the converse is false.

(e) Prove that if K/F and L/F are linearly disjoint finite Galois extensions then $\mathrm{Gal}(KL/F) \cong \mathrm{Gal}(K/F) \times \mathrm{Gal}(L/F)$.

2. Let K/F be a finite extension of fields and $f(x) \in F[x]$ an irreducible polynomial.

(a) Prove that if $\deg(f)$ does not divide $[K : F]$ then $f(x)$ has no roots in K.

(b) Prove that if $\deg(f)$ is relatively prime to $[K : F]$ then $f(x)$ is irreducible over K.

3. Let F be a field, $f(x)$ a polynomial over F with splitting field E/F.

(a) Let K/F be a subextension of E/F. Prove that E/K is a splitting field of $f(x)$ considered as a polynomial over K.

(b) Prove that if $\deg(f) = n$ then $[E : F]$ divides $n!$. (We only had $[E : F] \leq n!$ before.)
Hint. Use induction on n, and deal with cases of f reducible or irreducible separately. At some point you'll need the fact that $a!b!$ divides $(a+b)!$, which you should also prove.

4. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Determine the $\mathbb{Q}$-automorphism group $\mathrm{Aut}_{\mathbb{Q}} K$ by writing down all the elements as automorphisms and also by describing the isomorphism class of the group.

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