Dartmouth College Department of Mathematics
Math 71 Algebra
Fall 2021

Problem Set # 3 (due via Canvas upload by 5 pm, Wednesday, October 20)

Notation: Given a subset A of a group G, the **subgroup generated by** A is the subset $\langle A\rangle$ in G of all products of powers of elements in A, which is actually a subgroup of G. The main result of DF 2.4 is that $\langle A\rangle$ coincides with the intersection of all subgroups of G that contain A, in other words, $\langle A\rangle$ is the "smallest" subgroup of G containing A.

Reading: DF 2.2–2.5, 3.1–3.2.

Problems:

1. DF 2.2 Exercises 7*, 12, 14.

2. DF 2.4 Exercises 6, 7, 8, 9* (You already know how to compute the order of $mathrmSL_2(\mathbb{F}_3)$, so do it!), 11* (Hint: What are the orders of elements in S_4?), 12*, 13, 14*, 15, 19.

3. DF 2.5 Exercises 4, 10, 12*, 14*, 15.

4. DF 3.1 Exercises 5–12, 14, 17*, 22, 34, 36*, 40, 41*, 42.

5. DF 3.2 Exercises 4*, 5, 8*, 9, 13*, 16*, 22* (Euler's theorem!).

6. Show that for all $n, m \geq 1$, the group S_{n+m} contains a subgroup isomorphic to $S_n \times S_m$. Conclude that $n!m!$ divides $(n+m)!$.

7. *Tricks with Euler's theorem.* You can only use pencil and paper!

(a) Prove that every element of $(\mathbb{Z}/72\mathbb{Z})^\times$ has order dividing 12. (Hint: This is better than what a straight application of Euler's theorem will give you! Try applying Euler's theorem to a pair of relatively prime divisors of 72.)

(b) Find the last two digits of the huge number $3^{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}$ where there are 2021 threes appearing! (Hint: Do nested applications of Euler's theorem.)

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