Yale University Department of Mathematics
Math 612 Arithmetic & Geometry of Linear Algebraic Groups
Spring 2015

Final Exam (due 5:30 pm Thursday May 14th)

Guidelines. You may use any external sources, but please do not work together.

Notations. Let k be an arbitrary field. For an affine algebraic k-group scheme G, denote by $\underline{\mathrm{Aut}}(G) : \mathsf{Alg}_k \to \mathsf{Group}$ the functor of automorphisms. For the groups we have been considering, this functor is representable by an affine k-group scheme. Let $\underline{\mathrm{Inn}}(G)$ be the subgroup scheme of inner automorphisms. For a tensor (V, t), denote by $\underline{\mathrm{Aut}}(V, t) : \mathsf{Alg}_k \to \mathsf{Group}$ the group scheme of automorphisms of V preserving t. This is a closed subgroup scheme of $\mathrm{GL}(V)$. Write M_n for the algebra of $n \times n$ matrices.

1. Some basic constructions with central simple algebras of degree 2.

(a) The *classical adjoint* $\alpha : M_2 \to M_2$ defined by

$$\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is a k-linear determinant-preserving involution of symplectic type. Prove that $\mathrm{SL}_2 = \mathrm{Sp}(M_2, \alpha)$. Recall that the *symplectic group* $\mathrm{Sp}(A, \sigma)$ of an algebra (A, σ) with symplectic involution has group of R-points $\{x \in A \otimes_k R : x\sigma(x) = 1\}$ for $R \in \mathsf{Alg}_k$. Prove that $\mathrm{PSL}_2 \cong \mathrm{PGL}_2$ is isomorphic to the group scheme $\underline{\mathrm{Aut}}(M_2, \alpha)$ of algebra automorphisms of M_2 preserving α.

(b) Let A be a central simple algebra of degree 2 over k. Recall the *reduced norm* $\mathrm{N}_{A/k} : A \to k$ and *reduced trace* $\mathrm{T}_{A/k} : A \to k$. Define the *standard involution* σ_A of A by $\sigma_A(x) = \mathrm{T}_{A/k}(x)\, 1_A - x$. Prove that $x\sigma_A(x) = \mathrm{N}_{A/k}(x)$ for $x \in A$. (Hint: Remember that the reduced norm and trace were defined as coefficients of the characteristic polynomial of x in A.) Prove that σ_A is an involution of symplectic type on A. (Hint: Extending scalars so that A becomes isomorphic to M_2, what does σ_A become?) In fact, it's the unique such involution!

2. In this problem you will give a classification of forms of the semisimple linear algebraic group $\mathrm{SL}_2 \times \mathrm{SL}_2$.

(a) Compute the automorphism group scheme $\underline{\mathrm{Aut}}(\mathrm{SL}_2 \times \mathrm{SL}_2)$. Clearly, the map s switching the factors of SL_2 is an outer automorphism. Are there others?

(b) Prove that $\underline{\mathrm{Aut}}(\mathrm{SL}_2 \times \mathrm{SL}_2)$ is isomorphic to the group scheme $\underline{\mathrm{Aut}}(M_2 \times M_2)$ of algebra automorphisms of $M_2 \times M_2$. (Hint: How does an algebra automorphism of $M_2 \times M_2$ restrict to its center?)

(c) Conclude that the set of isomorphism classes of forms of $\mathrm{SL}_2 \times \mathrm{SL}_2$ are in bijection with set of isomorphism classes of pairs (A, K) where K is an étale quadratic algebra of degree 2 over k and A is a central simple algebra of degree 2 over K. Given (A, K) what is the associated form of $\mathrm{SL}_2 \times \mathrm{SL}_2$? Use the inner automorphism exact sequence

$$1 \to \underline{\mathrm{Inn}}(\mathrm{SL}_2 \times \mathrm{SL}_2) \to \underline{\mathrm{Aut}}(\mathrm{SL}_2 \times \mathrm{SL}_2) \to \underline{\mathrm{Out}}(\mathrm{SL}_2 \times \mathrm{SL}_2) \to 1$$

to describe the inner forms. Notice that the map $H^1(k, \underline{\mathrm{Inn}}(\mathrm{SL}_2 \times \mathrm{SL}_2)) \to H^1(k, \underline{\mathrm{Aut}}(\mathrm{SL}_2 \times \mathrm{SL}_2)) = \mathrm{Forms}(\mathrm{SL}_2 \times \mathrm{SL}_2)$ is not injective!

(d) Describe the forms when $k = \mathbb{R}$.

3. In this problem, you can assume, for simplicity, that the characteristic of k is $\neq 2$. Consider the determinant map as a quadratic form $\det : M_2 \to k$ on the space of 2×2 matrices. Let $\mathrm{SO}(M_2, \det)$ be its special orthogonal groups. Over $\mathbb{R}$, this is called $\mathrm{SO}_{2,2}$.

(a) Prove that the classical adjoint $\alpha : M_2 \to M_2$ defines an element of $\mathrm{O}(M_2, \det)$ that is not in $\mathrm{SO}(M_2, \det)$. Conclude that (conjugation by) α generates the outer automorphism group $\underline{\mathrm{Out}}(\mathrm{SO}(M_2, \det))$.

(b) Show that the map

$$\mathrm{SL}_2 \times \mathrm{SL}_2 \longrightarrow \mathrm{SO}(M_2, \det)$$

defined on R-points by $(A, B) \mapsto (X \mapsto AXB^{-1})$ for $R \in \mathsf{Alg}_k$, is a central isogeny with kernel the diagonally embedded μ_2. Conclude that this map yields an isomorphism of group schemes $\mathrm{SL}_2 \times \mathrm{SL}_2 \cong \mathrm{Spin}(M_2, \det)$.

(c) Prove that the above map induces an isomorphism

$$\underline{\mathrm{Aut}}(\mathrm{SL}_2 \times \mathrm{SL}_2) \longrightarrow \underline{\mathrm{Aut}}(\mathrm{SO}(M_2, \det)).$$

First, show that the diagonal μ_2 is fixed by all automorphisms of $\mathrm{SL}_2 \times \mathrm{SL}_2$; this induces a homomorphism $\underline{\mathrm{Aut}}(\mathrm{SL}_2 \times \mathrm{SL}_2) \to \underline{\mathrm{Aut}}(\mathrm{SO}(M_2, \det))$. (Hint: To show this is an isomorphism, match up the inner automorphism exact sequences for the two groups; on the level of outer automorphisms, verify that the switch map is taken to conjugation by the classical adjoint.) As an interesting side note, conclude that the explicit map $\varphi \mapsto \big(X \mapsto \varphi(X^t)^t\big)$ is an outer automorphism of $\mathrm{SO}(M_2, \det)$.

(d) Describe the resulting bijection (from taking nonabelian H^1)

$$\mathrm{Forms}(\mathrm{SL}_2 \times \mathrm{SL}_2) \longrightarrow \mathrm{Forms}(\mathrm{SO}(M_2(k), \det))$$

using our previous description of forms of $\mathrm{SL}_2 \times \mathrm{SL}_2$ and the description of forms for a special orthogonal group given in class in terms of central simple algebras (here of degree 4) with orthogonal involution.

You will need the following construction. Given (A, K) as before, let τ be the nontrivial element of the Galois group of K/k, and define ${}^{\tau}A$ to be the same underlying k-algebra as A but with the K-action twisted by τ. The naive switch map on $A \otimes_K {}^{\tau}A$ is a τ-semilinear algebra automorphism. Define the *algebra norm* $\mathrm{N}_{K/k}A$ of A from K down to k to be the k-subalgebra of elements of $A \otimes_K {}^{\tau}A$ invariant under the naive switch map. This turns out to be a central simple algebra of degree 4 over k. You should verify that restricting the involution $\sigma_A \otimes \sigma_{{}^{\tau}A}$ from $A \otimes_K {}^{\tau}A$ to $\mathrm{N}_{K/k}A$ yields an involution of orthogonal type. This verification can be performed over the separable closure of k, where K is split and A becomes isomorphic to $M_2 \times M_2$. When $K = k \times k$ and $A = A' \times A''$, then this construction yields $(A' \otimes_k A'', \sigma_{A'} \otimes \sigma_{A''})$.

(e) Finally, describe this bijection when $k = \mathbb{R}$. Recall that a quadratic form over $\mathbb{R}$ is uniquely determined up to isometry by its dimension and signature.

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