Yale University Department of Mathematics
Math 608 Introduction to Arithmetic Geometry
Fall 2018
Final Exam (due 5:30 pm Wednesday December 19th)

Guidelines. You may use any external sources, but work by yourself.

Notations. A **global field** is a finite extension of either $\mathbb{Q}$ or $\mathbb{F}_q(t)$. A quadratic form q over a field K is called **isotropic** if it has a nontrivial zero defined over K, and otherwise, is called **anisotropic**.

1. Let G be a finite cyclic group of order n and fix a generator σ. Let A be a G-module (i.e., abelian group with G-action). Consider the maps $N : A \to A$ and $\sigma - 1 : A \to A$ defined by

$$N(x) = \sum_{i=0}^{n-1} \sigma^i(x) \qquad \text{and} \qquad (\sigma - 1)(x) = \sigma(x) - x.$$

(a) Verify that the $\mathbb{Z}[G]$-module $\mathbb{Z}$ has a free resolution

$$\cdots \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ is the usual **augmentation** map or **counit** sending every group element to 1. It might help to learn about homotopy retractions.

(b) Show that this resolution gives the following periodicity on the level of cohomology

$$H^0(G, A) = A^G \qquad \text{and} \qquad H^i(G, A) = \begin{cases} {}_N A/(\sigma - 1)A & \text{if } i \text{ is odd} \\ A^G/NA & \text{if } i \text{ is even} \end{cases}$$

for $i > 0$, where ${}_N A = \ker(N : A \to A)$.

(c) Give formulas for $H^i(G, A)$ when G acts trivially on A.

2. Let L/K be a finite Galois extension with cyclic Galois group G.

(a) Use the cohomology of cyclic groups to show that the cohomological form of Hilbert's Theorem 90, namely $H^1(G, L^\times) = 1$, is equivalent to the classical form: that $x \in L^\times$ satisfies $N_{L/K}(x) = 1$ if and only if $x = \sigma(y)/y$ for some $y \in L^\times$.

(b) Recall that $\mathrm{Br}(L/K) = \ker(\mathrm{Br}(K) \to \mathrm{Br}(L))$. Use the cohomology of cyclic groups to prove that $\mathrm{Br}(L/K) \cong K^\times/N_{L/K}(L^\times)$.

(c) Let K have characteristic $\neq 2$ and $L = K(\sqrt{a})$ a quadratic extension of K. Prove that every 2-torsion element of $\mathrm{Br}(L/K)$ is represented by a quaternion algebra of the form (a, b) for some $b \in K^\times$ and that $(a, b) \cong (a, b')$ if and only if $b'/b = x^2 - ay^2$ for $x, y \in K$. Explicitly classify the 2-torsion elements in $\mathrm{Br}(\mathbb{Q}(i)/\mathbb{Q})$.

3. Let K be a field of characteristic 0 with algebraic closure $\overline{K}$. Assume that the absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ is cyclic of prime order p.

(a) Prove that $\mathrm{Br}(K) \cong K^\times/K^{\times p}$.
Hint. Use the long exact sequence in Galois cohomology associated to the Kummer sequence, along with Hilbert's Theorem 90, and the cohomology of cyclic groups.

(b) Conclude that $N_{\overline{K}/K}(\overline{K}^\times) = K^{\times p}$ and hence that the only possibility is $p = 2$ and $\overline{K} = K(\sqrt{-1})$.
Hint. Show that K contains a primitive pth root of unity (if not try adjoining it), hence that the cyclic extension $\overline{K}/K$ is a Kummer extension.

(c) Show that declaring the squares in $K^\times$ to be positive will equip K with the structure of an ordered field.

(d) (Artin–Schreier) Prove that if K is a field of characteristic 0 whose absolute Galois group is a nontrivial finite group, then $\overline{K} = K(\sqrt{-1})$ and K is an ordered field where the squares are positive.
Hint. Take a p-Sylow subgroup of the Galois group and use the fact that p-groups are solvable, then iteratively apply the previous results.
Remark. Such fields are called **real closed**. In fact, Artin and Schreier proved that in positive characteristic, the absolute Galois group is either trivial or infinite.

4. Let K be a global field of characteristic $\neq 2$ and Ω_K its set of places.

(a) Let q be a nondegenerate quadratic form in 3 variables over K. Prove that if q is isotropic over K_v for all but possibly a single $v \in \Omega_K$, then q is isotropic over K.

(b) Let q be a nondegenerate quadratic form in 4 variables over K with square discriminant. Prove that if q is isotropic over K_v for all but possibly a single $v \in \Omega_K$, then q is isotropic over K.

(c) Consider the quadratic form $q = x_1^2 + x_2^2 + x_3^2 + 7x_4^2$ defined over $\mathbb{Q}$. Prove that q is isotropic over $\mathbb{Q}_p$ for every prime p, but that q is anisotropic over $\mathbb{R}$.