Yale University Department of Mathematics
Math 370 Fields and Galois Theory
Spring 2018

Midterm Exam 1 Review Sheet

Directions: The first midterm exam will take place in class on Thursday, February 22. You will have the entire class period, 75 minutes, to complete the exam. No electronic devices will be allowed. No notes will be allowed. On all problems, you will need to write your thoughts/proofs in a coherent way to get full credit.

Topics covered and practice problems:

- Fields. Field extensions. Finite extensions. Degree. Power law. Algebraic and transcendental extensions. Finitely generated fields. Quadratic formula.
- Polynomial rings $F[x]$ and $\mathbb{Z}[x]$. Euclidean division. Euclidean algorithm. Irreducible polynomials. Ideal theory. Unique factorization.
- Irreducibility criteria. Reduction modulo p. Quadratic and cubic polynomials and their discriminants (formulas provided). Gauss's lemma. Primitive polynomials. Eisenstein criterion. Irreducible polynomials over $\mathbb{F}_p$.
- Classification of simple extensions. Minimal polynomial.
- Splitting fields.
- Compass and straightedge. Constructable points. Pythagorean closure.

Practice exam questions:

1. For which prime numbers $p \leq 20$ is the polynomial $x^2 + x + 1$ irreducible over $\mathbb{F}_p$?

2. Prove that the following polynomials are irreducible over $\mathbb{Q}$: $x^2 + 3x - 45$, $x^3 - x + 56$, $3x^4 + 6x + 125$, $x^6 - 5x^3 + 1$, $15x^7 + 12x^5 + 10x^3 + 8x - 6$, $7x^7 - 6x^6 + 4x^4 - 2x^2 + x - 21$. **Hint.** For the 5th one, search in Wikipedia for "Reciprocal polynomial."

3. Prove that the ideal generated by $x^2 - x + 1$ and 5 in $\mathbb{Z}[x]$ is a maximal ideal.

4. Factor $x^5 + 1$ over the field $\mathbb{F}_2$.

5. Determine whether the roots of the polynomial $x^4 - 5x^2 + 1$ are algebraic over $\mathbb{Q}$ and/or real and/or constructible.

6. For each of the following polynomials over $\mathbb{Q}$, determine the degree (over $\mathbb{Q}$) of its splitting field, and a set of at most two generators: $2x^2 + 3x + 4$, $x^3 - 21x - 28$, $x^6 - 1$, $x^7 - 1$.

7. Find a simple generator for the field extensions $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})/\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2}, \sqrt[3]{3})$.

8. Compute the minimal polynomial over $\mathbb{Q}$ of the following algebraic numbers:
$\sqrt{2}$, $1 + \sqrt{2}$, $\sqrt{1 + \sqrt{2}}$, $1 + \sqrt{1 + \sqrt{2}}$, $\sqrt{1 + \sqrt{1 + \sqrt{2}}}$.

9. Let $\pi \in \mathbb{R}$ be the area of a unit circle and let $\alpha = \sqrt{\pi^2 + 2}$. Consider the field $K = \mathbb{Q}(\pi, \alpha)$. For the following field extensions, determine whether they are transcendental and/or algebraic and/or finite and/or simple, and if you determine the extension is simple and algebraic, find a simple generator and determine its minimal polynomial:
$K/\mathbb{Q}$, $K/\mathbb{Q}(\pi)$, $K/\mathbb{Q}(\alpha)$, $K/\mathbb{Q}(\pi + \alpha)$.