Yale University Department of Mathematics
Math 370 Fields and Galois Theory
Spring 2018

Problem Set # 8 (due in class on Thursday April 5)

Notation: You can use the fact, which we will prove later, that a finite extension is separable and normal if and only if it is Galois.

Reading: GT 9, 17.4–17.5.

Problems:

1. Let K/F be a finite extension of fields of characteristic $p > 0$. Prove that $\alpha \in K$ is separable over F if and only if $F(\alpha) = F(\alpha^p)$. **Hint.** If $\alpha \in K$ is inseparable over F, find a nonzero $F(\alpha^p)$-derivation on $F(\alpha)$ to deduce something useful about the extension $F(\alpha)/F(\alpha^p)$. Conversely, what would the minimal polynomial of $\alpha \in K$ be over $F(\alpha^p)$.

2. Let K/F be an algebraic extension of fields of characteristic $p > 0$. Prove that the following are equivalent.

(a) Every element $\alpha \in K \setminus F$ is inseparable over F.

(b) For every $\alpha \in K$, there exists $n \geq 1$ such that $\alpha^{p^n} \in F$.

We call such extensions K/F **purely inseparable**. Warning: Just because $\alpha \in K$ is inseparable over F, it does not mean that every element of $F(\alpha)$ is inseparable over F. You might try to even find an example just to make sure!

3. Let K/F be a finite extension of fields of characteristic $p > 0$. Prove that K/F is purely inseparable if and only if $K = F(\alpha_1, \ldots, \alpha_n)$ and for each $1 \leq i \leq n$ there exists $n_i \geq 1$ such that $\alpha_i^{p^{n_i}} \in F$. **Remark.** We might call the elements $\alpha \in K$ that satisfy condition (b) in Problem 2 "purely inseparable" elements. In comparison to the warning in Problem 2, an extension generated by *purely* inseparable elements is actually *purely* inseparable.

Prove that $\mathbb{F}_p(t)[x]/(x^p - t)$ is a purely inseparable extension of $\mathbb{F}_p(t)$ of degree p.

4. Let K/F be a finite extension. Prove that there exists an intermediate extension $K/M/F$ such that M/F is separable and K/M is purely inseparable. **Hint.** Use the condition (b) in Problem 2 to construct M.

5. Let K/F be a finite Galois extension, and F'/F be any extension. Let $K' = K.F'$ be the compositum of K and F'. Prove that K'/F' is a Galois extension whose Galois group is isomorphic to a subgroup of $\mathrm{Gal}(K/F)$.

6. Let $\gamma = \sqrt{2 + \sqrt{2}} \in \mathbb{R}$.

(a) Show that $\mathbb{Q}(\gamma)/\mathbb{Q}$ is Galois with cyclic Galois group.

(b) Show that $\mathbb{Q}(\gamma, i) = \mathbb{Q}(\zeta_{16})$ and is Galois over $\mathbb{Q}$.

7. The 12th roots of unity. Let $\zeta = \zeta_{12}$.

(a) Prove that $x^4 - x^2 + 1$ is the minimal polynomial of ζ over $\mathbb{Q}$ and that the other zeros are $\zeta^5, \zeta^7, \zeta^{11}$.

(b) Prove that $\mathbb{Q}(\zeta)/\mathbb{Q}$ is a Galois extension and that there is an isomorphism of groups

$$(\mathbb{Z}/12\mathbb{Z})^\times \to \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$$
$$j \mapsto \left(\varphi_j : \zeta \mapsto \zeta^j\right)$$

so that the Galois group is a Klein four group.