Yale University Department of Mathematics
Math 370 Fields and Galois Theory
Spring 2018

Problem Set # 5 (due in class on Thursday March 1)

Notation: Let K and L be subfields of a field M. The **compositum** of K and L, denoted KL, is defined to be the smallest subfield of M containing both K and L, equivalently, the intersection of all subfields of M containing K and L. If additionally K and L are both extensions of a field F, we say that the extensions K/F and L/F are **linearly disjoint** if any F-linearly independent subset of K is L-linearly independent in KL and if any F-linearly independent subset of L is K-linearly independent in KL.

Reading: GT 7,8.

Problems:

1. GT Exercise 7.7, 7.19.

2. About the constructibility of regular n-gons.

(a) Let p be a prime number. Prove that if a regular p-gon can be constructed, then p must be of the form $p = 2^n + 1$. **Hint.** Use ζ_p.

(b) For each $3 \leq n < 17$, determine whether a regular n-gon can be constructed.

3. Let F be a field and K/F and L/F be subextensions of a field extension M/F.

(a) Prove that if $K = F(\alpha_1, \dots, \alpha_n)$ and $L = F(\beta_1, \dots, \beta_m)$ are finitely generated, then $KL = F(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m)$.

(b) Prove that if K/F and L/F are finite then KL/F is finite and $[KL : F] \leq [K : F]\,[L : F]$ with equality if and only if K/F and L/F are linearly disjoint.
Hint. Prove that if $x_1, \dots, x_n$ is an F-basis for K and $y_1, \dots, y_m$ is an F-basis for L, then the products $x_i y_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ span KL/F and are an F-basis if and only if K/F and L/F are linearly disjoint.

(c) Prove that finite extensions K/F and L/F of relatively prime degree are linearly disjoint. (Now you can do Problem 5a on Midterm 1 easily!)

(d) Let $f(x)$ be an irreducible polynomial over F and K/F a finite extension. Prove that if $\deg(f)$ and $[K : F]$ are relatively prime, then $f(x)$ is still irreducible over K.
Hint. Use the previous part.

(e) Prove that if K/F and L/F are linearly disjoint then $K \cap L = F$. Find an example showing that the converse is false.

4. Let F be a field, $f(x)$ a polynomial over F with splitting field K/F.

(a) Let L/F be a subextension of K/F. Prove that K/L is the splitting field of $f(x)$ considered as a polynomial over L.

(b) Prove that if $\deg(f) = n$ then $[K : F]$ divides $n!$. (Recall this was claimed in lecture.)
Hint. Use induction on n, and deal with cases of f reducible or irreducible separately. At some point you'll need the fact that $a!b!$ divides $(a + b)!$, which luckily you already proved in Math 350 Problem Set # 3.

5. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Determine the Galois group of $K/\mathbb{Q}$.

6. Let F be a field of characteristic $\neq 2$. Let $f(x) = x^4 + bx^2 + c \in F[x]$ with splitting field K/F. Assume that the roots of f are distinct. Prove that the Galois group of K/F is isomorphic to a subgroup of the dihedral group D_8 of order 8.