Yale University Department of Mathematics
Math 370 Fields and Galois Theory
Spring 2018

Problem Set # 3 (due in class on Thursday 8 February)

Reading: GT 4, 5, 6.

Problems:

1. GT Exercise 5.2.

2. GT Exercise 5.6. The point is to construct these extensions as subfields of $\mathbb{C}$. Just saying "$\mathbb{Q}[x]/(f(x))$ is a simple extension of $\mathbb{Q}$ generated by an element with minimal polynomial $f(x)$" is true, but is a bit cheating.

3. GT Exercise 5.7.

4. Factor $x^3 + x + 1$ in $\mathbb{F}_p$ for $p = 2, 3, 5$.

5. Let $a \in \mathbb{Q}$ be positive and not a square. Prove that $\mathbb{Q}(\sqrt[4]{a})$ has degree 4 over $\mathbb{Q}$.
Be careful here. It is usually not so bad to find an upper bound for the degree, but proving that the degree actually equals this bound often takes more work.

6. Let F be a field. We know that if $K = F(\alpha)$ and α has minimal polynomial $m(x)$, then $m(x)$ has a root over K. So over K, we can factor $m(x) = (x - \alpha)n(x)$. What more can we say in general about the factorization of $m(x)$ over K, i.e., how does $n(x)$ factor further?

(a) Let $\alpha = \cos(2\pi/9) \in \mathbb{R}$. Prove that α is algebraic over $\mathbb{Q}$ and find its minimal polynomial $m(x)$. (**Hint.** Try taking $e^{2\pi i/9}$ to various small powers.) Prove that $m(x)$ factors into linear factors over $\mathbb{Q}(\alpha)$. (**Hint.** There is nothing fancy going on here, just use the division algorithm to divide $m(x)$ by $x - \alpha$ over the field $\mathbb{Q}(\alpha)$. The quotient will have degree 2, so there's a formula you know for determining if it has roots; in trying to apply this formula, you may need to use a computer algebra package to solve a system of Diophantine equations.)

(b) Let $\alpha = \sqrt[3]{2} \in \mathbb{R}$. Prove that α is algebraic over $\mathbb{Q}$ and find its minimal polynomial $m(x)$. Prove that $m(x)$ does not factor into linear factors over $\mathbb{Q}(\alpha)$.

7. GT Exercises 6.8.

8. GT Exercise 6.9, 6.11.

9. GT Exercise 6.10. You might try doing 6.14 first.