

Homework #1 Solutions (9/18/06)

Chapter 1 Matrix Operations

3.12 Let A and D be square matrices (say $n \times n$ and $m \times m$, respectively), then

$$\det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = (\det A)(\det D),$$

where B is any $n \times m$ matrix and 0 is the $m \times n$ zero matrix.

Proof. Most people proved this by induction on the total size of the block matrix. I'll give an alternate way. We'll need the following special case as a preliminary lemma.

Lemma 1 Let A be an $n \times n$ matrix and I be the $m \times m$ identity matrix, then

$$\det \begin{pmatrix} A & B \\ 0 & I \end{pmatrix} = \det A,$$

where B is any $n \times m$ matrix.

Proof. This follows by induction and a expanding along the last row. For the interested readers, here is the formal proof by induction. Note that only $m \geq 1$ makes sense here. We'll do induction on m .

Base case: $m = 1$. In this case, for any $n \times n$ matrix A , expansion along the last row shows that

$$\det \begin{pmatrix} & & & b_1 \\ & A & & \vdots \\ & & & b_n \\ 0 & \cdots & 0 & 1 \end{pmatrix} = (-1)^{(n+1)+(n+1)} \det A = \det A.$$

Induction step. Suppose the lemma holds for $m \geq 1$, then by expanding along the last row, we have

$$\det \begin{pmatrix} A & B \\ 0 & I_{m+1} \end{pmatrix} = \det \begin{pmatrix} A & B' & B'' \\ 0 & I_m & 0 \\ 0 & 0 & 1 \end{pmatrix} = (-1)^{(n+m+1)+(n+m+1)} \det \begin{pmatrix} A & B' \\ 0 & I_m \end{pmatrix} = \det A,$$

where in the above, we break $B = (B' \ B'')$ into an $n \times m$ matrix B' and an $n \times 1$ matrix B'' , and where we use the induction hypothesis for the final equality. This is what we wanted to prove, so by induction, the lemma holds for all $m \geq 1$. \square

We note that the similar statement

$$\det \begin{pmatrix} I & B \\ 0 & D \end{pmatrix} = \det D,$$

can be proved in the same way (this time expanding along the first column) for any $m \times m$ matrix D and $n \times n$ identity matrix I .

OK, now on to the proof. First we note that if $\det(A) = 0$ then A is not invertible, hence the columns of A are linearly dependent. But then the first n columns of the block matrix are linearly dependent using the same linear combination (this uses the fact that the lower left block is zero). Hence the block matrix is not invertible, hence has determinant 0. So the formula we want holds if A

is not invertible. Thus we can assume that A is invertible. In that case, we have

$$\begin{aligned}\det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} &= \det \left(\begin{pmatrix} A & 0 \\ 0 & I_m \end{pmatrix} \begin{pmatrix} I_n & A^{-1}B \\ 0 & D \end{pmatrix} \right) \\ &= \det \begin{pmatrix} A & 0 \\ 0 & I_m \end{pmatrix} \det \begin{pmatrix} I_n & A^{-1}B \\ 0 & D \end{pmatrix} \\ &= (\det A)(\det B),\end{aligned}$$

where at the first line we use block multiplication of block matrices, at the second line the multiplicativity of the determinant, and at the final line our first lemma. You might say that this proof was longer than a straight out proof by induction, however, you could have gotten away with saying the first lemma is “obviously by expanding along the last row/first column” (because it is). I gave this proof here mostly for flavor. \square

***3.13** Let A, B, C , and D be $n \times n$ matrices. Suppose that A is invertible and $AC = CA$. Then we have

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB).$$

Proof. Since A is invertible, consider the following matrix product

$$\begin{pmatrix} A^{-1} & 0 \\ -C & A \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & A^{-1}B \\ -CA + AC & AD - CB \end{pmatrix} = \begin{pmatrix} I & A^{-1}B \\ 0 & AD - CB \end{pmatrix},$$

since $AC - CA = 0$. Now applying determinant to both sides, we have

$$\det \begin{pmatrix} A^{-1} & 0 \\ -C & A \end{pmatrix} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det \begin{pmatrix} I & A^{-1}B \\ 0 & AD - CB \end{pmatrix} = \det(AD - CB),$$

the final equality coming from our previous lemma 1. But now note that

$$\begin{aligned}\det \begin{pmatrix} A^{-1} & 0 \\ -C & A \end{pmatrix} &= \det \begin{pmatrix} A^{-1} & 0 \\ -C & A \end{pmatrix}^t = \det \begin{pmatrix} (A^{-1})^t & -C^t \\ 0 & A^t \end{pmatrix} \\ &= (\det(A^{-1})^t)(\det A^t) = (\det A^{-1})(\det A) \\ &= \det(A^{-1}A) = \det I = 1,\end{aligned}$$

where the first equality is from the fact that determinant is transpose invariant and looking at what the transpose does to a block matrix, the third line is from the multiplicativity of the determinant and the fact that the determinant of the identity is 1. So in the end our previous equality is what we were looking for. \square

***M.18** Let A be an $m \times n$ matrix and B be an $n \times m$ matrix. Then $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible.

Proof. Here’s the trick. First note that we have a generalization of 3.13 in the following form

$$\det \begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} = \det(I_m - AB).$$

This follows from considering the following matrix product

$$\begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} \begin{pmatrix} I_m & 0 \\ -B & I_n \end{pmatrix} = \begin{pmatrix} I_m - AB & A \\ 0 & I_n \end{pmatrix},$$

and as before taking the determinant of both sides

$$\det \begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} \det \begin{pmatrix} I_m & 0 \\ -B & I_n \end{pmatrix} = \det \begin{pmatrix} I_m - AB & A \\ 0 & I_n \end{pmatrix},$$

which gives what we want since the determinant of the second matrix from the left is 1 by 3.12 (combined with the transpose trick at the end of 3.13) and the determinant of the matrix on the right is $\det(I_m - AB)$ again from 3.12.

Now by swapping $n \cdot m$ rows and then $n \cdot m$ columns we see that

$$\det \begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} = (-1)^{nm+nm} \det \begin{pmatrix} I_n & B \\ A & I_m \end{pmatrix},$$

and so by the above formula (switching the roles of m and n to apply it the right hand side of the last equation) we arrive at

$$\det(I_m - AB) = \det(I_n - BA),$$

from which we see that one side is nonzero if and only if the other side is nonzero, which was what we wanted to prove. \square