Yale University Department of Mathematics
Math 350 Introduction to Abstract Algebra
Fall 2017

Problem Set # 7 (due at the beginning of class on November 3)

Notation: If a group G acts on a set of n elements, then the permutation representation gives a homomorphism $\rho : G \to S_n$. Keep this in mind!

Reading: DF 4.5, 4.6, and 5.1.

Problems:

1. DF 4.5 Exercises 18, 22, 26, 30, 34*, 36*, 39, 40.

2. DF 4.6 Exercise 1.

3. DF 5.1 Exercises 4, 14*, 15, 17.

4. *Solvable up to sixty!* Recall that A_5, which has order 60, is simple. The goal of this problem is to prove that all groups of order < 60 are solvable.

($\aleph$) Prove that if 60 is the first order of a finite nonabelian simple group, then all groups of order < 60 are solvable. Hint: Jordan–Hölder.

As the abelian simple groups are those of prime order, for each *composite* order $n < 60$, we will try to prove that no group of order n is simple. For example, we already know that no group of order p^α, with $\alpha > 1$, is simple and that no group of order pq, with p and q primes, is simple. Prove the following additional criteria on the order of a group for the group to not be simple:

(a) If G is a finite group of order $p^k m$, with $p \nmid m$ and $m < p$ (more generally, no divisor of m other than 1 is congruent to 1 modulo p), then G has a normal Sylow p-subgroup.

(b) If G is a finite group of order pqr, where p, q, and r are primes with $p < q < r$, then G has a normal Sylow subgroup for at least one of p, q, or r.

(c) If G is a finite group of order $2^k \cdot 3$, with $k \geq 1$, then G is not simple.

(d) If G is a finite group of order $2^k \cdot 5$, with $k \geq 1$, then G is not simple.

(e) If G is a finite group of order $2^2 \cdot 3^k$, with $k \geq 1$, then G is not simple. For $k = 1$, use part (c).

(f) If G is a finite group of order $3^k \cdot 5$, with $k \geq 1$, then G is not simple.

Hints: The first follows from a direct application of the congruence conditions in the Sylow theorems. For the second, assume the contrary and consider the possible number of Sylow r-subgroups, then use this to count the number of elements of order r (any two Sylow r-subgroups intersect only at the identity), combine this with the number of elements of order p and q to find more elements than the order of the group. For the third and fourth, handle k small using the congruence conditions and for k large, consider the permutation representation associated to the conjugation action of G on the Sylow 2-subgroups. For the fifth and sixth, do the same using the Sylow 3-subgroups.

(g) From the above criteria, prove that every group whose order is < 60 and is neither prime nor 56, is not simple.

(h) No group of order 56 is simple.

Hint: If neither the Sylow 2- nor 7-subgroups are normal, start counting elements in these subgroups to reach a contradiction (while any two Sylow 7-subgroups only intersect at the identity, how could Sylow 2-subgroups intersect?).

Great, you done! For fun, how much higher can you go using these same tools?

(i) (Optional.) What is the largest number $N > 60$ you can find such that no group whose order is composite and between 61 and N is simple?