Yale University Department of Mathematics
Math 350 Introduction to Abstract Algebra
Fall 2017

Problem Set # 5 (due at the beginning of class on Friday 13 October)

Notation: If G is a group then the set of automorphisms, i.e., isomorphisms $\varphi : G \to G$, is a group under composition called the **automorphism group** of G and denoted $\mathrm{Aut}(G)$. For every $g \in G$, conjugation by g is an automorphism of G, and all these conjugation automorphisms form a subgroup of $\mathrm{Aut}(G)$ called the **inner automorphism group** of G and denoted $\mathrm{Inn}(G)$.

By the "isomorphism type" of a group, we mean all groups that are isomorphic to that group. By "identify the isomorphism type" of a group G, we mean "give another group that is friendly to you, whose name you know, that is isomorphic to G". I know this sounds like a vague notion, but it is quite helpful. For example, "cyclic of order n", "Klein four", "S_n" are all descriptions of isomorphism types of groups.

Reading: DF 3.3, 3.4, 4.1, 4.2, 4.4.

Problems:

1. DF 3.3 Exercises 2, 3, 6*, 7* (note that in particular, if also $M \cap N = \{e\}$ then $G \cong M \times N$ and we say that G is the **internal direct product** of M and N), 8, 9*.

2. DF 3.5 Exercises 10* (A_4 is the subgroup of S_4 generated by 3-cycles).

3. DF 3.4 Exercises 2, 4*, 7, 8* (Hint: in (i), use the definition of solvable given in DF; to prove (iii) implies (iv), use the hint given in DF and not the method I was trying to give in office hours, the idea is to build up a sequence of subgroups inductively from scratch, where the hint is describing the smallest subgroup in the sequence).

4. DF 4.1 Exercises 3*, 4, 6, 7*, 8, 9.

5. DF 4.2 Exercises 2, 10*, 11, 14.

6. *Finite vector spaces.* Let V be an $\mathbb{F}_p$-vector space of (finite) dimension n.

(a) What is the isomorphism type of the underlying finite abelian group $(V, +)$?

(b) Show that the automorphism group $\mathrm{Aut}((V, +))$ of the abelian group $(V, +)$ is isomorphic to the group $\mathrm{GL}(V)$ of $\mathbb{F}_p$-linear vector space isomorphisms $\varphi : V \to V$ and that this group is also isomorphic to $\mathrm{GL}_n(\mathbb{F}_p)$.

(c) Compute the order of $\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$. Find an automorphism of order 7.

Yale University, Department of Mathematics, 10 Hillhouse Ave, New Haven, CT 06511
E-mail address: asher.auel@yale.edu