Yale University Department of Mathematics
Math 350 Introduction to Abstract Algebra
Fall 2017

Problem Set # 4 (due at the beginning of class on Friday 6 October)

Reading: DF 3.1, 3.3, 3.5.

Problems:

1. DF 3.1 Exercises 5–12, 14*, 17*, 22, 25, 32, 34, 36*, 40, 41*, 42.

2. DF 3.2 Exercises 4*, 5, 14*, 19, 21* (Hint: try dividing by an integer relatively prime to the index of this purported subgroup).

3. *More classification.* Prove that if G is an abelian group of order pq, where p an q are distinct primes numbers, then G is cyclic. You can use Cauchy's theorem for abelian groups.

4. *Some isomorphisms.*

(a) For any field F, prove that the center of $\mathrm{GL}_2(F)$ consists of $F^\times$ multiples of the identity matrix. What is the center of $\mathrm{SL}_2(F)$? We denote by $\mathrm{PGL}_2(F) = \mathrm{GL}_2(F)/Z(\mathrm{GL}_2(F))$ and $\mathrm{PSL}_2(F) = \mathrm{SL}_2(F)/Z(\mathrm{SL}_2(F))$.

(b) Prove that $\mathrm{GL}_2(F)$ acts on the set P of lines in F^2 through the origin and that the kernel of this action is the center of $\mathrm{GL}_2(F)$. Here, "line through the origin" is a colloquial term for "1-dimensional subspace." Conclude that $\mathrm{PGL}_2(F)$ acts faithfully on the set P, hence the permutation representation is an injective homomorphism $\mathrm{PGL}_2(F) \to S_P$ to the symmetric group on the elements of P.

(c) Calculate $|\mathrm{PGL}_2(\mathbb{F}_p)|$.

(d) Prove that $\mathrm{PGL}_2(\mathbb{F}_3) \cong S_4$. (Hint: How many lines through the origin are there in $\mathbb{F}_3^2$?)

(e) Under the isomorphism $\mathrm{PGL}_2(\mathbb{F}_3) \cong S_4$ from the previous part, find all elements of $\mathrm{PGL}_2(\mathbb{F}_3)$ corresponding to 3-cycles.

(f) For an odd prime p, prove that the map $\mathrm{PSL}_2(\mathbb{F}_p) \to \mathrm{PGL}_2(\mathbb{F}_p)$, taking the coset represented by a matrix M to the coset represented by M, is a well defined injective homomorphism whose image has index 2. Notice that for $p = 3$ this is particularly clear!

(g) Prove that the determinant yields a well-defined homomorphism $\det : \mathrm{PGL}_2(\mathbb{F}_3) \to \mathbb{F}_3^\times$. Show that $\mathrm{PSL}_2(\mathbb{F}_3) = \ker(\det)$.

(h) Show that $\mathrm{PSL}_2(\mathbb{F}_3)$ is isomorphic to the subgroup of S_4 generated by the 3-cycles.

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