Yale University Department of Mathematics
Math 350 Introduction to Abstract Algebra
Fall 2017

Extra Fun Problems

Notation: A_n denotes the alternating group on n objects, the index 2 subgroup of the symmetric group S_n, generated by even products of 2-cycles. For example, A_4 is the subgroup of S_4 generated by 3-cycles.

Recall the definition of $\mathrm{PGL}_2(F)$ and $\mathrm{PSL}_2(F)$ from Problem Set #4.

Problems:

1. Prove that an integer $n > 1$ is a prime number if and only if $(n-1)! \equiv -1 \pmod{n}$.

2. Prove the following presentations.

(a) $A_4 = \langle x, y \mid x^2 = y^3 = (xy)^3 = 1 \rangle$

(b) $S_4 = \langle x, y \mid x^2 = y^3 = (xy)^4 = 1 \rangle$

(c) $A_5 = \langle x, y \mid x^2 = y^3 = (xy)^5 = 1 \rangle$

3. Recall the set of lines through the origin in $\mathbb{F}_q^2$.

(a) Prove that $\mathrm{PGL}_2(\mathbb{F}_4) \cong \mathrm{PSL}_2(\mathbb{F}_4) \cong \mathrm{SL}_2(\mathbb{F}_4) \cong A_5$.

(b) Prove that $\mathrm{PGL}_2(\mathbb{F}_5) \cong S_5$ and that $\mathrm{PSL}_2(\mathbb{F}_5) \cong A_5$. Hint: The action on the lines gives an injective homomorphism $\mathrm{PGL}_2(\mathbb{F}_5) \to S_6$. Count the number of $(2,2,2)$-cycles in S_6 that are not in the image of this homomorphism, then let $\mathrm{PGL}_2(\mathbb{F}_5)$ act on them by conjugation to give a new permutation representation.

4. *Some linear algebra over the field of order 9.*

(a) Prove that $\mathbb{F}_3[i] = \{0, \pm 1, \pm i, \pm 1 \pm i\}$, where $i^2 = -1$ and all other arithmetic is done modulo 3, is a field of order 9, which we call $\mathbb{F}_9$.

(b) Prove that $\mathrm{PGL}_2(\mathbb{F}_9)$ is *not* isomorphic to S_6, even though they have the same order. Hint: Use some linear algebra to bound the size of the conjugacy classes in $\mathrm{PGL}_2(\mathbb{F}_9)$ and compare with what we'll learn about S_6.

(c) Prove, on the other hand, that $\mathrm{PSL}_2(\mathbb{F}_9) \cong A_6$. Hint: Find a subgroup of $\mathrm{PSL}_2(\mathbb{F}_9)$ isomorphic to A_5, and then act on the 6 cosets to get a permutation representation.

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